On the Dynamic Properties of the Labor-Surplus Economy

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Abstract

The dynamic properties of the optimal growth model are examined, based on a one good and two factor-labor and capital--model with a labor-surplus economy due to a fixed wage rate. If the average capital productivity is higher than the time discount rate which is assumed to be larger than the population growth rate, then the economy reaches the full employment stationary state in a finite amount of time. If both are equal, then the economy stays in the initial state forever. If the average capital productivity is less than the population growth rate due to a high and fixed wage rate, then the economy is in the poverty trap where both per capita capital and per capita consumption shrink to zero. If the average productivity is higher than population growth rate but lower than time discount rate, there can exist two different paths – one to stationary state and the other on the poverty trap – depending on the initial values of per capita capital if the elasticity of marginal felicity is less than one. Therefore the only way to get out of the poverty trap is to increase average capital productivity above the time discount rate indirectly by increasing labor productivity by means of a big push or imports of technology and/or by increasing per capita capital through imports of capital.

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I. Introduction

This paper shows the dynamic properties of the one sector growth economy with surplus labor due to a fixed wage rate. To our best knowledge, Lewis' (1954) paper is the first to attract attention by explaining the existence of a labor-surplus economy based on the limited demand for labor given a higher wage rate in the urban sector on the one hand, and the unlimited supply of labor given a lower subsistence wage level in the rural sector. Dixit (1968) investigates the dynamic properties of Lewis's static model, assuming laborers never save and capitalists save a portion of their surplus which is equal to output less wage payments. Marglin (1976) employs model similar to Dixit's, but it differs from Dixit's in the sense that the time discount rate is zero in Marglin's model and the instantaneous felicity function is the one of constant elasticity of marginal utility – larger than unity.

To some extents we follow Dixit's (1968) model, except that we assume neither a specific saving function, nor that laborers do not save and only capitalists save a portion of their surplus. On the other hand, in order to obtain an analytically solvable problem we specify an instantaneous felicity function to be of constant elasticity of marginal felicity σ as assumed by Marglin.

We assume that there exists one good which is used either for consumption or investment and produced employing labor and capital with a neoclassical production function. Capital depreciation is assumed away for simplicity. The population growth rate, *n*, is constant. The wage rate is fixed so that a labor surplus is persistent. Then the capital-labor ratio is fixed as well. With these settings we maximize the discounted sum of the instantaneous felicity function over time subject to the law of motion of capital.

Then we can reduce the problem to solve the second order homogenous ordinary differential equation in per capita capital *k* with constant coefficients. We obtain the following results; first under $\sigma < 1$ if the average capital productivity $\overline{p}$ is higher than the time discount rate ρ, (ρ is assumed to be higher than n) then the economy reaches the stationary state in a finite time where full employment is maintained thereafter. Second, if the average capital productivity is less than the population growth rate, then the economy converges to the stationary state when both per capita capital and per capita consumption become zero as time goes to infinity (the case of the vicious circle of poverty)[1/]. Third, if both are equal, then the economy stays in the initial state forever. (Proposition 1) However if $\overline{p}$ is less than ρ but higher than n, then the economy can get out of the poverty trap by increasing per capita capital or increasing $\overline{p}$. (Proposition 2). If $\sigma \geq 1$, then essentially the same results as

Proposition 1 holds. (Propositions 3 and 4)

From these results, we can recommend that in the labor surplus economy the average capital productivity should be increased indirectly by increasing labor productivity by means of a "big push"(i.e., a synchronized expansion in many factors.)[/2] or imports of technology[3/] and/or by importing capital. In fact, Nurkse (1955) pointed out this policy as a fundamental remedy to solve the poverty of underdeveloped countries due to the lack of the size of markets. In the next section we develop our model. In the last section we conclude briefly.

II Model

We follow the notation of Dixit (1968) as much as possible. Output Y is produced using capital K and Labor L;

$$Y = F(K, L) \tag{1}$$

where F is a neoclassical production function; both marginal products of capital and labor are positive, and F is concave and homogenous of degree one in (K, L).

Output is used either for consumption C or investment I;

$$Y = C + I \tag{2}$$

Capital increases by investment;

$$\dot{K} = I, \tag{3}$$

where $\dot{K}$ is the derivative of K with respect to time t. We assume away depreciation of capital for simplicity.

Population N increases by the rate of n so that

$$\dot{N}/N = n. \tag{4}$$

Let lower case variables denote the corresponding upper case variables per unit of population N;

Let $u(c)$ be the instantaneous felicity function of per capita consumption $c = c/N$ whose elasticity of marginal felicity σ is constant;

$$u(c) = \begin{cases} \dfrac{1}{1-\sigma} c^{1-\sigma}, & 0 < \sigma \neq 1 \\ \log c, & \sigma = 1 \end{cases}.$$

Let $h = K/L$ and $l = L/N$ where h is the capital-labor ratio and $1-l$ is the unemployment rate with $0 \leq l \leq 1$. We assume both the labor and capital markets to be perfectly competitive. Now let w and r respectively be the wage rate and rental price of capital so that

$$F_L(K,L) = w \quad \text{and} \quad F_K(K,L) = r.$$

Let $F(\cdot,\ 1) = f(\cdot)$. Then

$F_L(K,L) = F_L(K/L,\ 1) = f(h) - hf'(h) = w$. As the wage rate w is fixed if and only if h is fixed, we denote $h = \bar{h}$ to be fixed. This is the main difference between our model, and Dixit(1968)'s and Marglin(1976)'s model. In their models, the wage rate is fixed, but not necessarily equal to the marginal products of labor. We also note r to be fixed. Since

$$y = Y/N = (L/N)(Y/L) = lf(h) = lf(\bar{h}) = l\bar{v}$$

where $f(\bar{h}) = \bar{v}$ is fixed. Then

$$k = K/N = (K/L)(L/N) = \bar{h}l \le \bar{h}\text{, i.e.,}$$

$$k \le \bar{h} \tag{5}$$

holds. We obtain from (1), (2), (3) and (4)

$$\dot{k} = k\bar{p} - c - nk \tag{6}$$

where $\dot{k} = dk/dt$ is the derivative of k with respect to time t, and $y = l\bar{v} = k\bar{v}/\bar{h} = k\bar{p}$,

and $\bar{p} = \bar{v}/\bar{h} = f(\bar{h})/\bar{h}$ is the average capital productivity. Now we investigate the following utility maximization problem;

$$\max \int_0^\infty u(c)e^{-(\rho-n)t}dt$$

subject to (5) and (6) where $t = 0$ is the present time. Then by letting the Hamiltonian H be;

$$H = u(c) + q(k\bar{p} - c - nk) + \lambda(\bar{h} - k)$$

we obtain the following first order conditions

$$c \ge 0,\ c(c^{-\sigma} - q) = 0 \ \text{ and } \ c^{-\sigma} = q \ \text{ if } \ c > 0 \tag{7}$$

$$\dot{q} = \rho q - q\bar{p} + \lambda \tag{8}$$

$$\lambda \ge 0 \ \text{ and } \ \lambda(\bar{h} - k) = 0, \tag{9}$$

and the transversality condition

$$\lim_{t\to\infty} qke^{-(\rho-n)t} = 0\,. \tag{10}$$

Equations (8) and (9) show the characteristics of the labor-surplus economy. With unemployment, $\lambda = 0$ and the economy is reduced to the conventional one. With full employment realized in a finite time T (as shown later) at the stationary state, $\lambda = \bar{\lambda} > 0$, $k = \bar{h}$ and $c = \bar{c} > 0$ hold. Notably, c stops increasing suddenly at $t = T$ (i.e., $\dot{c}$ the increasing rate of c in time, becomes discontinuous at $t = T$). We make the following assumption throughout for the existence of the optimal solution;

We assume throughout

A. 1 $0 < k_0 < \bar{h}$,

i.e., there exists unemployment initially.

Stationary State

Here we consider the stationary state where $\dot{k} = \dot{c} = \dot{q} = 0$ holds. As seen from (11) below, there exist two stationary states depending on the sign $\beta = (\overline{p} - \rho)/\sigma$. First we consider the case of $\beta > 0$.

By letting $k = \overline{k}$ be the stationary value of *k* (variables with – denote their stationary values.) and $\dot{k} = k\overline{p} - c - nk = 0$, we obtain $\overline{c} = \overline{k}(\overline{p} - n)$. Furthermore from (7), $\overline{q} = \overline{c}^{-\sigma}$, and $\overline{\lambda} = (\overline{p} - \rho)\overline{q} > 0$ follow from (8) with $\dot{q} = 0$. These imply that if $\overline{k} = \overline{h}$ holds under $\overline{p} > \rho$ ($\beta = (\overline{p} - \rho)/\sigma > 0$), then the economy reaches the stationary state where full employment is obtained in a finite time $t = T(< +\infty)$ from $\overline{c} = \overline{k}(\overline{p} - n) > 0$ and (11) shown below.

However if $\overline{k} < \overline{h}$ holds under $\overline{p} < \rho$ ($\beta < 0$), then the economy is on the poverty trap where both *k* and *c* decrease monotonically to zero as time goes to infinity, and $\overline{\lambda} = 0$ and $\overline{k} = \overline{c} = 0$ follow from $\dot{k} = (\overline{p} - n)k - c = 0$ and $\dot{c} = \sigma^{-1}(\overline{p} - \rho - \lambda c^{\sigma})c$ $= 0$ with $\lambda = 0$. That is, $\overline{k} = \overline{c} = \overline{\lambda} = 0$ is another stationary state.($\overline{q} < +\infty$ follows from $\overline{q} < c^{-\sigma} = \infty$.) (If $\overline{p} - n = 0$ holds, then $k = k_0$ and $c = c_0$ follow where k_0 and c_0 are respectively the initial values of *k* and *c*. That is, the economy stays in the initial state forever.) Here we note that regarding the transversality condition (10), since $\dot{q}/q + \dot{k}/k - (\rho - n) = \lambda/q - c/k$ holds from (6) and (8), in the case of $\overline{\lambda} = (\overline{p} - \rho)\overline{q}$ > 0 and $\overline{c} = \overline{k}(\overline{p} - n) = \overline{h}(\overline{p} - n) > 0$, $\lambda/q - c/k = \overline{\lambda}/\overline{q} - \overline{c}/\overline{k} = \overline{p} - \rho - (\overline{p} - n)$ $= -(\rho - n) < 0$ at $t = \infty$, and in the case of $\lambda = \overline{\lambda} = 0$ and $\overline{c} = \overline{k} = 0$, $\dot{q}/q + \dot{k}/k - (\rho - n) = -c/k = -c_0/k_0 = -(\alpha - \beta) < 0$ holds where $\alpha = \overline{p} - n$ and $\beta = (\overline{p} - \rho)/\sigma$ as shown below.

In short for both cases where a stationary state is reached as time goes to infinity, the transversality condition is always satisfied. Here we note from the definition of $\overline{p}$, $\overline{p} = f(\overline{h})/\overline{h}$ and $\overline{w} = f(\overline{h}) - \overline{h}f'(\overline{h})$, that $\overline{p}$ is higher if and only if $\overline{w}$ is lower. However in the labor-surplus economy due to high and fixed wage rate $\overline{w}$, $\overline{p}$ is lower not due to the capital-scarcity but to the wage rigidity. Then as explained later the only viable optimal path is the so called vicious circle of poverty trap where both per capita capital and per capita consumption decrease to zero as the equilibrium is approached.

Next we consider the case where $k < \overline{h}$ holds, i.e., the economy is not in the stationary state.

Then from (7) and (8) recalling $\lambda = 0$ from (9), we obtain $\dot{c} = \sigma^{-1}c(\overline{p} - \rho)$. Henceforth we assume without loss of generality $c_0 > 0$, i.e., initial per capita consumption is positive. Then we obtain

$$c = c_0 e^{(\overline{p}-\rho)\sigma^{-1}t}. \tag{11}$$

Next by further differentiating (6) with respect to time *t*, we obtain from (6) and (11)

$$\ddot{k} = \overline{p}\dot{k} - \dot{c} - n\dot{k} = \overline{p}\dot{k} + \sigma^{-1}(\rho - \overline{p})c - n\dot{k}$$
$$= \overline{p}\dot{k} + \sigma^{-1}(\rho - \overline{p})(\overline{p}k - nk - \dot{k}) - n\dot{k},$$

or

$$\ddot{k} - (\alpha + \beta)\dot{k} + \alpha\beta = 0 \tag{12}$$

where $\alpha = \overline{p} - n$ and $\beta = (\overline{p} - \rho)/\sigma$. Then recalling (12) is the second order homogenous ordinary differential equation in *k* with constant coefficients, we obtain for $\alpha \neq \beta$, *k* can be expressed as

$$k = Ae^{\alpha t} + Be^{\beta t},$$

where *A* and *B* are determined by

$$A + B = k_0, \tag{13}$$
$$\alpha A + \beta B = \overline{p}k_0 - c_0 - nk_0. \tag{14}$$

Here k_0 and c_0 are the initial values of *k* and *c*, i.e., $k_0 = k(0)$ and $c_0 = c(0)$ ($k = k(t)$ and $c = c(t)$).

Then

$A = k_0 - c_0/(\alpha - \beta)$ and $B = c_0/(\alpha - \beta)$ are derived.

In short

$$k = (k_0 - c_0/(\alpha - \beta))e^{\alpha t} + (c_0/(\alpha - \beta))e^{\beta t} \tag{15}$$

holds for $\alpha \neq \beta$. For $\alpha = \beta$, we obtain the following expression for *k*;

$$k = (k_0 - c_0 t)e^{\alpha t} \text{ [4/].} \tag{16}$$

As for the sgns of *α* and *β*, we obtain the following subcases;

(i) $\overline{p} > \rho$ (i.e., α and $\beta > 0$), (ii) $\overline{p} = \rho$ (i.e., $\alpha > 0$ and $\beta = 0$), (iii) $n < \overline{p} < \rho$ (i.e., $\beta < 0 < \alpha$), (iv) $\overline{p} = n$ (i.e., $\alpha = 0$ and $\beta < 0$), (v) $\overline{p} < n < \rho$ (i.e., α and $\beta < 0$).

Fig.1 The area of (*α*, *β*)

First we consider the case of $0 < \sigma < 1$.

There exist seven cases as for the combinations of α and β under $1 > \sigma > 0$;

(1) $0 < \alpha < \beta$, (2) $0 < \alpha = \beta$, (3) $0 < \beta < \alpha$, (4) $0 = \beta < \alpha$, (5) $\beta < 0 < \alpha$,
(6) $\beta < 0 = \alpha$ and (7) $\beta < \alpha < 0$ as shown in Fig. 1.

As seen below, there exist two possible paths of (k, c) for each (1), (2) and (3), i.e., for $\beta > 0$; one converges to the full employment stationary state in a finite time $t = T < +\infty$, while the other is on the vicious circle of poverty, i.e., both per capita capital k and per capita consumption c tend to zero as time goes to infinity.
However the former's utility is higher than that of the latter irrespective of the initial value of k, k_0, implying that the economy converges to the full employment equilibrium independent of the initial value of k as far as $\beta > 0$, i.e., the capital productivity $\bar{p}$ is higher than the time discount rate ρ.

For (4) $0 = \beta < \alpha$, the optimal path is to stay in the initial state, i.e., $c = c_0 = \alpha k_0$ and $k = k_0$. (5) $\beta < 0 < \alpha$ is a very interesting case where p is less than ρ but larger than n. Per capita consumption c keeps decreasing for a while, but depending on the initial value of k, the economy can converges to the full-employment stationary state. This is analyzed in Proposition 2 independently. Lastly for (6) and (7) where $\beta < 0$ and $\alpha \le 0$ hold, the optimal path is on the vicious circle of poverty.

In the following we show the above results for each case as Propositions 1 and 2 when $0 < \sigma < 1$ holds. We analyze the properties of the optimal paths for $\sigma \ge 1$ also for 3 cases both when $\sigma = 1$ and $1 < \sigma$ hold, and classify the results in Proposition 3 and 4 based on signs of α and β.

We employ the following system of the two dimensional ordinary equations

$$\begin{cases} \dot{k} = \alpha k - c & (6) \\ \dot{c} = \beta c & (17) \end{cases}$$

when (17) is obtained from (7) and (8) recalling $\lambda = 0$ if $k < \bar{h}$.

First we consider the case (1) $0 < \alpha < \beta$ under $\sigma < 1$.

Fig. 2

Let $t = T < \infty$ be the time at which the economy reaches the stationary state $E(\bar{h}, \bar{c})$. Then we can express the value function as

$$U = \max \int_0^\infty u(c) e^{-(\rho-n)t} dt = (1-\sigma)^{-1} \left\{ c_0^{1-\sigma} \int_0^T e^{at} dt + \bar{c}^{1-\sigma} \int_T^\infty e^{-bt} dt \right\}, \tag{18}$$

in view of (11) where $a = \beta - \alpha$ and $b = \rho - n > 0$. U is the value function obtained by the path (k, c) converging to the stationary state $E(\bar{h}, \bar{c})$ shown by the dotted line toward E in Fig. 2. Here

$$\bar{c} = \alpha \bar{h} = c_0 e^{\beta T} \tag{19}$$

holds from (11), implying $c_0 = \alpha \bar{h} e^{-\beta T}$.

Then we observe

$$(\alpha\bar{h})^{\sigma-1}(1-\sigma)U = e^{-(1-\sigma)\beta T}(e^{aT}-1)/a + e^{-bT}/b$$
$$= (e^{-bT} - c^{-bT-aT})/a + e^{-bT}/b = \{(1-e^{-aT})/a + 1/b\}e^{-bT} \tag{20}$$

from

$$(1-\sigma)\beta = \beta - \alpha + \rho - n = a + b\,. \tag{21}$$

Here we note that there exists another path converging toward the vertical axis as shown in Fig. 2. We have to show that the value function obtained from this latter path is less than the one obtained from (20), i.e., from the optimal path converging to *E*. Let $t = \bar{t}$ be the time at which *k* becomes zero on the latter path. Then $\bar{t}$ is obtained implicitly from (15) with $k = 0$ as

$$\bar{t} = (\beta-\alpha)^{-1}\log(1 + k_0(\beta-\alpha)/c_0)\,. \tag{22}$$

Furthermore recalling both $k = \bar{h}$ and $\dot{k} = 0$ hold at $t = T$, we obtain

$$k_0 = c_0(\alpha e^{\alpha T} - \beta e^{\beta T})/e^{\alpha T}\cdot\alpha(\alpha-\beta) = \alpha\bar{h}e^{-\beta T}(\alpha e^{\alpha T} - \beta e^{\beta T})/e^{\alpha T}\cdot\alpha(\alpha-\beta)\,,$$

in short

$$k_0 = \bar{h}a^{-1}(\beta e^{-\alpha T} - \alpha e^{-\beta T})\,. \tag{23}$$

This is a critical equation by which the comparison between the two paths of (*k*, *c*) can be made. That is to say, we express the value functions U and $\tilde{U}$ respectively corresponding to the optimal path and the one converging toward the vertical axis as function of *T*, and show *U* to be higher than $\tilde{U}$.

Let $\tilde{U}$ be the value of the value function obtained from the path (*k*, *c*) converging to the horizontal axis at $t = \bar{t}$. Then we calculate

$$\tilde{U} = (1-\sigma)^{-1}\int_0^{\bar{t}} c_0^{1-\sigma}e^{at}dt = (1-\sigma)^{-1}c_0^{1-\sigma}(e^{a\bar{t}}-1)/a$$
$$= (1-\sigma)^{-1}c_0^{1-\sigma}k_0c_0 = (1-\sigma)^{-1}c_0^{-\sigma}k_0 \tag{24}$$

from (11) and (22). Clearly $\tilde{U}$ is maximized by taking $c_0 = \alpha k_0$ from $c_0 \geq \alpha k_0$.

Then we note from (22), $\bar{t} = (\beta-\alpha)^{-1}\log\beta/\alpha$ to be positive and finite, and hence $\bar{k} = \bar{c} = 0$ at $t = \bar{t}$, implying *c* becomes discontinuous at $t = \bar{t}$. Furthermore the transversality condition; Hamiltonian $H = 0$ is met at $t = \bar{t}$ from $0 < \sigma < 1$. [5/] Hence from (23) $\tilde{U}$ is expressed as

$$\tilde{U} = (1-\sigma)^{-1}\alpha^{-\sigma}k_0^{1-\sigma} = (1-\sigma)^{-1}\alpha^{-\sigma}\bar{h}^{1-\sigma}a^{\sigma-1}(\beta e^{-\alpha T} - \alpha e^{-\beta T})^{1-\sigma}$$
$$= (1-\sigma)^{-1}\alpha^{-\sigma}\bar{h}^{1-\sigma}a^{\sigma-1}\alpha^{1-\sigma}e^{-(1-\sigma)\beta T}(\frac{\beta}{\alpha}e^{aT}-1)^{1-\sigma}$$
$$= (1-\sigma)^{-1}(\alpha\bar{h})^{1-\sigma}a^{\sigma-1}\alpha^{-\sigma}e^{-bT-aT}(\frac{\beta}{\alpha}e^{aT}-1)^{1-\sigma} \tag{25}$$

from (23). Then from (20) and (25) we obtain

$$(1-\sigma)(\alpha h)^{\sigma-1}e^{bT}(U-\widetilde{U})=(1-e^{-aT})/a+1/b-a^{\sigma-1}\alpha^{-\sigma}e^{-aT}(\frac{\beta}{\alpha}e^{aT}-1)^{1-\sigma}.$$

Let $f(x)=(1-x^{-1})/a+1/b-a^{\sigma-1}\alpha^{-\sigma}x^{-1}(\frac{\beta}{\alpha}x-1)^{1-\sigma}$ where $x=e^{aT}\geq 1$. Then

$$U>\widetilde{U}>0 \Leftrightarrow f(x)>0 \qquad \forall x\geq 1, \tag{26}$$

which is shown in Appendix I. (26) shows that in case of $\beta>\alpha>0$, irrespective of the initial value of k_0, the optimal path of (k, c) is realized by choosing c to be $c=c_0e^{\beta t}=\alpha\bar{h}e^{\beta(t-T)}$ where T is implicitly defined by (23).

Next we consider case (2) $0<\alpha=\beta$. Noting for $\alpha=\beta$

$$\dot{k}=(\alpha k_0-\alpha c_0 t-c_0)e^{\alpha t}$$

holds from (16),

$$t=T=k_0/c_0-1/\alpha>0 \tag{27}$$

is derived. Recalling $a=\beta-\alpha=0$ and $\bar{c}=\alpha\bar{h}$, we see that the value function U is expressed as

$$U=(1-\sigma)^{-1}\left\{\int_0^T c_0^{1-\sigma}dt+(\alpha\bar{h})^{1-\sigma}\int_T^\infty e^{-bt}dt\right\}. \tag{28}$$

Furthermore from (16), $t=T$ is expressed as

$$T=(k_0-\bar{h}e^{-\alpha T})/c_0. \tag{29}$$

Then from (27), c_0 is expressed as

$$c_0=k_0/(T+1/\alpha),$$

and by substituting this into (29), we obtain

$$k_0=(\alpha T+1)\bar{h}e^{-\alpha T}. \tag{30}$$

Just as for case (1), there exist two possible paths of (k, c) as shown in Fig. 2; one converging to the stationary state $E(\bar{k},\bar{c})$ and the other converging toward the vertical axis in a finite time $\bar{t}=k_0/c_0$ from (10). We calculate the value functions for both. The one for the former path is expressed as (25) and by rearranging, it is rewritten as

$$(1-\sigma)U=c_0^{1-\sigma}T+(\alpha\bar{h})^{1-\sigma}e^{-bT}/b=(\alpha\bar{h})^{1-\sigma}(e^{-(1-\sigma)\alpha T}\cdot T+e^{-bT}/b)$$

from (19) with $\alpha=\beta$. The one for the latter path is expressed as

$$(1-\sigma)\widetilde{U}=\int_0^{\bar{t}}c_0^{1-\sigma}dt=c_0^{1-\sigma}\bar{t}=c_0^{-\sigma}k_0$$

from $\bar{t}=k_0/c_0$ with $c_0\geq\alpha k_0$. $\widetilde{U}$ is maximized by taking $c_0=\alpha k_0$, and is expressed as

$$(1-\sigma)\widetilde{U} = \alpha^{-\sigma} k_0^{1-\sigma} = \alpha^{-\sigma} \overline{h}^{1-\sigma} e^{-(1-\sigma)\alpha T} (\alpha T + 1)^{1-\sigma},$$

from (30). Here we note again at $t = \bar{t} = 1/\alpha$, k become zero and also the per capita consumption becomes zero suddenly, and hence the transversality condition $H = 0$ is satisfied. Then we obtain

$$(1-\sigma)(\alpha\overline{h})^{\sigma-1}(U - \widetilde{U}) = e^{-(1-\sigma)\alpha T} \cdot T + e^{-bT}/b - \alpha^{-1} e^{-(1-\sigma)\alpha T} (\alpha T + 1)^{1-\sigma}$$

$$= e^{-bT} \cdot T + e^{-bT}/b - \alpha^{-1} e^{-bT} (\alpha T + 1)^{1-\sigma}$$

noting $(1-\sigma)\alpha = b$ from (21). Hence

$$U > \widetilde{U} \Leftrightarrow T + 1/b - \alpha^{-1}(\alpha T + 1)^{1-\sigma} > 0 \quad {}^{\forall}T > 0$$

$$\Leftrightarrow \alpha(T + 1/b) > (\alpha T + 1)^{1-\sigma} \quad {}^{\forall}T > 0,$$

which holds always[6/]. Again as for case (1) $0 < \alpha < \beta$, we obtain, irrespective of the initial value of k_0, the path of (k, c) converging to $E(\overline{h}, \overline{c})$ is seen to be the optimal.

Next we consider case (3) $0 < \beta < \alpha$. Again just as the case (1), we obtain the phase diagram of Fig. 2, and that there exist two possible paths of (k, c); one converging to the stationary state $E(\overline{h}, \overline{c})$ and the other converging toward the vertical axis. Then similarly as for case (1), we can calculate the value functions for these two paths, U and $\widetilde{U}$ to obtain

$$(\alpha\overline{h})^{\sigma-1}(1-\sigma)U = (e^{-bT-aT} - e^{-bT})/(\alpha - \beta) + e^{-bT}/b$$

and

$$(\alpha\overline{h})^{\sigma-1}(1-\sigma)\widetilde{U} = (\alpha - \beta)^{\sigma-1} \alpha^{-\sigma} e^{-bT-aT} (1 - \frac{\beta}{\alpha} e^{aT})^{1-\sigma}.$$

Hence we note

$$U > \widetilde{U} \Leftrightarrow (e^{-aT} - 1)/(\alpha - \beta) + 1/b > (\alpha - \beta)^{\sigma-1} \alpha^{-\sigma} e^{-aT} \ (1 - \frac{\beta}{\alpha} e^{aT})^{1-\sigma} \quad {}^{\forall}T > 0,$$

which holds always.[7/]

Hence again for case (3), the path of (k, c) converging to the stationary state $E(\overline{h}, \overline{c})$ is seen to be optimal inspective the initial value of k.

Next we consider the cases (6) $\beta < 0 = \alpha$ and (7) $\beta < \alpha < 0$ altogether, i.e., for the cases $\beta < \alpha \leq 0$. Case (5) $\beta < 0 < \alpha$ is very interesting and seems to deserve to be analyzed separately later.

Fig. 3 **Fig. 4**

The phase diagram corresponding to cases (6) and (7) are drawn in Fig. 3.

The optimal path of (k, c) converges to the stationary state $E(0, 0)$, the origin.

Let $t = \bar{t}$ be the time at which *k* becomes zero, which is expressed by (22) as

$$\bar{t} = -(\alpha - \beta)^{-1} \log(1 - k_0 (\alpha - \beta)/c_0). \tag{22}$$

Then the corresponding value function $\widetilde{U}$ is given by (24) as

$$\widetilde{U} = (1-\sigma)^{-1} c_0^{-\sigma} k_0. \tag{24}$$

Here in view of (22), $c_0 \geq (\alpha - \beta)k_0$ must hold. Especially in case of $c_0 = (\alpha - \beta)k_0$, $\bar{t} = +\infty$ follows from (22). Cleary $\widetilde{U}$ is maximized by taking $c_0 = (\alpha - \beta)k_0$. These two cases reflect the vicious circle of poverty when both *k* and *c* decrease to zero as time goes to infinity. Here we note $c/k = c_0/k_0 = (\alpha - \beta)$ holds from (11) and (15) always.

Lastly we consider the case (4) $0 = \beta < \alpha$. The corresponding phase diagram is shown in Fig. 4. Since *c* is constant, $c = c_0$ follows. The time $t = \bar{t}$ at which *k* becomes zero is again given by (22) ($\bar{t}$ may be infinite), and hence the value function is expressed as

$$\widetilde{U} = (1-\sigma)^{-1} \int_0^{\bar{t}} c_0^{1-\sigma} e^{-bt} dt = (1-\sigma)^{-1} c_0^{1-\sigma} \, (1 - e^{-b\bar{t}})/b = (1-\sigma)^{-1} c_0^{-\sigma} \cdot k_0 \tag{31}$$

from (22) (noting $\alpha - \beta = \alpha = b$) which is maximized by taking $c_0 = (\alpha - \beta)k_0 = \alpha k_0$ and hence $\bar{t} = +\infty$. Then $k = k_0$ follows from (15).

Next we calculate the value function *U* when (4) $0 = \beta < \alpha$ and $c_0 < \alpha k_0$ hold so that $c = c_0$ till $t = T$ and $c = \bar{c} = \alpha \bar{h}$ from $t \geq T$.

That is,

$$U = (1-\sigma)^{-1} \left\{ \int_0^T c_0^{1-\sigma} e^{-bt} dt + \int_T^\infty \bar{c}^{1-\sigma} e^{-bt} dt \right\},$$

and hence

$$(1-\sigma)U = c_0^{1-\sigma} (1 - e^{-bT})/b + \bar{c}^{1-\sigma} e^{-bT}/b.$$

Here we obtain

$$e^{\alpha T} = (\bar{h} - c_0/\alpha)/(k_0 - c_0/\alpha)$$

from (15) with $\beta = 0$ and $k = \bar{h}$. By substituting this into the above, recalling $\alpha = b$ holds when $\beta = 0$, we obtain

$$b(1-\sigma)U = c_0^{1-\sigma}(1 - (k_0 - c_0/\alpha)/(\bar{h} - c_0/\alpha)) + \bar{c}^{1-\sigma}(k_0 - c_0/\alpha)/(\bar{h} - c_0/\alpha). \tag{32}$$

Here observing that the right hand side increases as c_0 increases[8/], we obtain *U* is maximized by taking $c_0 = \alpha k_0$ and hence $T = \infty$. Then maximized value of *U* is equal to $\widetilde{U} = (1-\sigma)^{-1} \alpha^{-\sigma} k_0^{1-\sigma}$, and hence we can conclude for (4) $0 = \beta < \alpha$, to let

$c = c_0 = \alpha k_0$ and $k = k_0$ always in the optimal path.

Fig. 5

Summing up the above arguments, we obtain

Proposition 1. Let $0 < \sigma < 1$,

(I) If $\beta > 0$, i.e., the average capital productivity $\overline{p}$ is higher than the time discount rate ρ, then irrespective of the initial value of per capita capital k_0, the optimal path of (c, k) is the one which increases monotonically to the full employment stationary state $E(\overline{h}, \overline{c})$ in a finite time and stay there thereafter. Initial per capita consumption c_0 is equal to $\alpha \overline{h} e^{-\beta T}$ where $t = T$ is the time at which the optimal path reaches the stationary state. (T is given implicitly by (22)).

(II) If $\beta = 0$, i.e., $\overline{p}$ is equal to ρ, then the optimal path (choice) is to stay in the initial level of per capita consumption and per capita capital, by taking $c = c_0 = \alpha k_0$ and $k = k_0$ always.

(III) If $\beta < \alpha \leq 0$, i.e., $\overline{p}$ is less than ρ and equal or less than n, then irrespective of k_0 the optimal path of (c, k) is on the vicious circle of poverty, i.e., it decreases monotonically toward the origin as time goes to infinity.

The proposition shows that the critical factor which decides the directions of the optimal path of (k, c) is not the level of the initial per capita capital k_0 but the sign of $\beta = (\overline{p} - \rho)/\sigma$. That is to say, irrespective of the level of k_0, if the average capital productivity $\overline{p}$ is higher than the time discount rate ρ, then the optimal path is the one moving toward the full employment stationary state $E(\overline{h}, \overline{c})$. However if $\overline{p}$ is less than ρ and equal or less than n, then the optimal path is on the vicious circle poverty trap, i.e., the one moving toward the origin.

Hence for the economy to get out of the poverty trap and move into the path toward the full employment equilibrium, it is fatal to improve the average capital productivity abve the level of time discount rate (perhaps indirectly by improving the labor productivity by various means).

Here we consider the case (5) $\beta < 0 < \alpha$ under $0 < \sigma < 1$. The phase diagram corresponding to this case is drawn in Fig. 5. There exist two paths of (k, c) depending on the initial state (k_0, c_0). The first is the one converging to the origin and starting from $c_0 \geq (\alpha - \beta) k_0$ — This inequality comes from (22) — as shown below, and the other is the one converging toward the vertical axis $k = \overline{h}$ in a finite time T and stays in the stationary state $E(\overline{h}, \overline{c})$ after T.

Just as in the case for (1) $0 < \alpha < \beta$, the value function $\widetilde{U}$ for the first path is expressed by (24) and the time $t=\bar{t}$ at which k vanishes is given by (22). Then $\tilde{U}$ is seen to be maximized by taking $c_0=(\alpha-\beta)k_0$ and hence $\bar{t}=\infty$, implying $\tilde{U}$ is given by

$$\tilde{U}=(1-\sigma)^{-1}(\alpha-\beta)^{-\sigma}k_0^{1-\sigma}.$$

On the other hand, the value function U corresponding to the second path is given by

$$U=(1-\sigma)^{-1}\left(\int_0^T c_0^{1-\sigma}e^{at}dt+\int_T^\infty \bar{c}^{1-\sigma}e^{-bt}dt\right)$$

$$=(1-\sigma)^{-1}\left\{c_0^{1-\sigma}(1-e^{aT})/(\alpha-\beta)+\bar{c}^{1-\sigma}e^{-bT}/b\right\}.$$

Here noting T is given implicitly by (15) with $k=\bar{h}$ so that

$$\bar{h}=(k_0-c_0/(\alpha-\beta))e^{\alpha T}+(c_0/(\alpha-\beta))e^{\beta T},$$

we obtain U to be maximized by taking $c_0=\alpha k_0$ and hence the maximized by U is expressed as

$$U=(1-\sigma)^{-1}\left\{(\alpha k_0)^{1-\sigma}(1-e^{aT})/(\alpha-\beta)+(\alpha\bar{h})^{1-\sigma}e^{-bT}/b\right\}.$$ (See Appendix II.)

Henceforth we assume
A. 2 for $\beta<0$ and $\alpha>0$ $\rho-n<\alpha((\alpha-\beta)/\alpha)^\sigma$ holds.
A. 2 holds if $\sigma(<1)$ is near 1.[9/]

Proposition 2. For (5) $\beta<0<\alpha$, and $0<\sigma<1$, under A. 2, there exists $k^*\in(0,\bar{h})$ such that $U<\tilde{U}$ if and only if $k<k^*$. (See Appendix II.)

The situation Proposition 2 assumes seems to reflect the reality of some developing countries where the per capita consumption c keeps decreasing ($\beta<0$, i.e., $p<\rho$), but the average capital productivity $\bar{p}$ is higher than the population growth rate $(\alpha>0)$, in short $n<\bar{p}<\rho$. Furthermore we assume that per capita capital k is below k*. In these situations, such countries can get out of the poverty trap by increasing per capita capital k above certain level k*. Although per capita consumption keeps decreasing after k becomes above k* for a certain time, k becomes increasing and reaches the level of $\bar{h}$ and then c jumps up to $\bar{c}=\alpha\bar{h}$ at the same time and stay there thereafter.

Next we consider the case of $\sigma=1$, i.e., $u(c)=\log c$. $\beta<\alpha$ holds always for $\sigma=1$. The value function for this felicity function for $\beta<0$ is expressed as

$$\tilde{U}=\int_0^{\bar{t}}u(c)e^{-(\rho-n)t}dt=(k_0/c_0)\log c_0-(\beta/b)\left\{-(1-bk_0/c_0)b^{-1}\log(1-bk_0/c_0)-k_0/c_0\right\}^{10/} \quad (31)$$

where $t=\bar{t}$ is given by

$$\bar{t} = -(\alpha - \beta)^{-1} \log(1 - bk_0 / c_0) . \tag{22}$$

As far as $\beta < \alpha \le 0$, the phase diagram is as drawn in Fig. 3. Here from $c_0 \ge bk_0$ (which is derived from (22)), we observe that $c_0 = bk_0$ must hold for the existence of the optimal path (which is on the poverty trap) and hence $\bar{t} = +\infty$. Otherwise i.e., $c_0 > bk_0$, *k* vanishes at $t = \bar{t} < +\infty$ form (22), and so does *c*, causing the violation of the transversality condition (the Hamiltonian is equal to zero at $c = 0$). Hence from $a = \beta - \alpha = -(\rho - n) = -b < 0$, $c / k = c_0 / k_0 = b = \alpha - \beta$ holds from (11) and (15) for $\sigma = 1$.Then $\widetilde{U}$ is expressed as

$$\widetilde{U} = (1/b) \log bk_0 + \beta / b^2 \tag{32}$$

noting $(1 - bk_0 / c_0) \log(1 - bk_0 / c_0) \to 0$ as $c_0 \to bk_0$ [11/].

Next we consider the case of $\beta < 0 < \alpha$ for $\sigma = 1$. Here we note that although there exist two paths of (*k*, *c*) for $\alpha > 0$ as seen from Fig. 5, the one starting from $c_0 \le \alpha k_0$ is not seen to be optimal. In fact for this path, the value function *U* is expressed as

$$U = \int_0^\infty (\log c) e^{-bt} dt = \int_0^T (\log c_0 + \beta t) e^{-bt} dt + \int_T^\infty (\log \bar{c}) e^{-bt} dt , \tag{33}$$

where *T* is implicitly defined by (15) as

$$\bar{h} = (k_0 - c_0 /(\alpha - \beta)) e^{\alpha T} + (c_0 /(\alpha - \beta)) e^{\beta T} . \tag{34}$$

U is maximized by taking $c_0 = \alpha k_0$ just as the case (5) $\beta < 0 < \alpha$ for $0 < \sigma < 1$. But *U* is seen to be less than $\widetilde{U}$ always. (See Appendix IV.)

Next we consider the case of $\beta = 0$. ($\alpha > 0$ follows from $\beta = 0 < \alpha$.) $c = c_0$ must hold always, which is made possible by taking $c_0 = \alpha k_0 = bk_0$, and hence $k = k_0$ from (15). If $c_0 > \alpha k_0$ holds initially, then $c = c_0$ till $t < \bar{t}$ and *c* becomes zero at $t = \bar{t} < +\infty$ from (22) as drawn in Fig. 4, since $k = 0$ at $t = \bar{t}$. But then Hamiltonian *H* does not vanish at $t = \bar{t}$, violating the transversality condition. On the other hand if $c_0 < \alpha k_0$ initially, then again as drawn in Fig. 4 $c = c_0$ till $t < T$ and $c = \bar{c}$ from $t \ge T$. Then the corresponding value function is expressed as

$$U = \int_0^\infty (\log c) e^{-bt} dt = \int_0^T (\log c_0) e^{-bt} dt + \int_T^\infty (\log \bar{c}) e^{-bt} dt$$

$$= (\log c_0)(1 - e^{-bT}) / b + (\log \bar{c}) e^{-bT} / b .$$

Here since c_0 is a function of *T* from (34) so that

$$c_0' = dc_0 / dT = (\alpha \bar{h} - c_0) e^{-\alpha T} \alpha / (1 - e^{at}) > 0$$

from $c_0 \le \alpha k_0 < \alpha \bar{h}$, by totally differentiating $f(T) = U$ with respect to T we obtain from $a = -\alpha = -b$,

$$f'(T) = (c_0'/c_0)(1-e^{-bT})/b + (\log c_0)e^{-bT} - (\log \bar{c})e^{-bT}$$

$$= (\alpha \bar{h} - c_0)e^{-bT}/c_0 + (\log c_0)e^{-bT} - (\log \bar{c})e^{-bT},$$

and hence

$e^{bT} f'(T) = (\alpha \bar{h} - c_0)/c_0 + \log c_0 - \log \bar{c} > 0$ [12/]. Hence $f(T)$ is maximized by taking $c_0 = \alpha k_0$ from $c_0 \le \alpha k_0$. Then $k = k_0$ holds from (15). In short for $\beta = 0$ and $\sigma = 1$, $c = c_0 = \alpha k_0$ and $k = k_0$ is the optimal path.

Lastly we consider the case of $\beta > 0$. Since $\beta < \alpha$ holds always for $\sigma = 1$, $0 < \beta < \alpha$ is the only possible combination of α and β for $\beta > 0$. Then we obtain the phase diagram as shown in Fig. 2. Here we note that although there exist two possible paths of (*k*, *c*) which satisfy the first order conditions, the one converging toward the vertical axis does satisfy the transversality condition. In fact as seen from Fig. 2, the initial value of *c* must satisfy $c_0 \ge \alpha k_0$. Then however from (21), $\bar{t} < +\infty$ follows from $b < \alpha$. Then recalling $c = 0$ holds at $t = \bar{t}$ from $k = 0$ at $t = \bar{t}$, we obtain that the transversality condition, $H = 0$ is violated. Then we observe there exists an optimal path of (*k*, *c*) with similar properties as before. Hence we observe

Proposition 3. Under $\sigma = 1$,
(I)if $\beta > 0$, then the optimal path of (*k*, *c*) is the one converging monotonically to $E(\bar{h}, \bar{c})$ with $c_0 = (\alpha \bar{h})e^{-\beta T}$ in a finite time $t = T$ defined by (22) and remains $E(\bar{h}, \bar{c})$ thereafter.
(II) If $\beta = 0$, *k* and *c* remains their initial values, i. e., $k = k_0$ and $c_0 = \alpha k_0 (= bk_0)$.
(III) If $\beta < 0$, then the path of (*k*, *c*) is on the poverty trap, i.e., (*k*, *c*) converges monotonically to the origin on time goes to infinity with $c_0 = bk_0$.

Proposition 3 is similar to Proposition 1 both for (I) $\beta > 0$ and (II) $\beta = 0$, i.e., for (I), the economy converges monotonically to the full employment equilibrium in a finite time and stay there thereafter. For (II), the economy stays in the initial state. The difference lies in (III) $\beta < 0$. For case of $\sigma < 1$ if $\beta < \alpha \le 0$, then the economy is on the poverty trap, while if $\beta < 0 < \alpha$, then as shown in Proposition 2 depending on the initial amount of capital either it is on the poverty trap (if initial capital is small) or on the path to the full employment equilibrium (if the initial capital is larger). However for $\sigma = 1$, as far as $\beta < 0$ the economy is on the poverty trap even if $\alpha > 0$ since the utility derived from the path to the full employment equilibrium is always less than that from the poverty trap.

Again just as Proposition 1, we note that which direction of approach is taken depends on the sign of β, but not on the value of initial per capita capital k_0.

Here we discuss the case of $1 < \sigma$. We first discuss the following case; $\alpha < \beta < 0$ (which follows from $1 < \sigma$ and $\bar{p} < n < \rho$. See Fig.1.) Recalling the system of ordinary equations (6) and (17), and its phase diagram is drawn as in Fig. 3, the optimal path of (*k*, *c*) is the one moving toward the origin and the time $t = \bar{t}$ at which *k* vanishes is given by (22). Then the corresponding value function is given by (24). Hence we observe that for a given finite value of c_0, *k* vanishes at $t = \bar{t} < +\infty$, and so does *c*. However then the Hamiltonian $H = u(c) + q(k\bar{p} - c - nk) + \lambda(\bar{h} - k)$ tends to $+\infty$ as $c \to 0$ with $t \to \bar{t} < +\infty$, violating the transversality condition of the free finite terminal problem of optimal control theory. Hence we must assume c_0 to be infinite for $\bar{t} < +\infty$, implying the non-existence of the optimal path. The same holds for $\alpha = \beta < 0$ since $\bar{t} = k_0 / c_0 < \infty$ holds from (16). There exist another five cases for the possible combinations of α and β under $1 < \sigma$; (a) $0 < \beta < \alpha$, (b) $0 = \beta < \alpha$, (c) $\beta < 0 < \alpha$, (d) $\beta < 0 = \alpha$ and (e) $\beta < \alpha < 0$ as seen from Fig. 1. For (d) and (e) as seen from Fig. 3, by taking $c_0 = (\alpha - \beta)k_0$, (*k*, *c*) is seen to decrease monotonically toward the origin as $t \to \infty$ from (22), (i.e., on the poverty trap.). For (a), (b) and (c), the paths of (*k*, *c*) converging either toward the origin or the vertical axis are seen to violate the transversality condition (i.e., $k = 0$ at $t = \bar{t} < +\infty$), and hence not to exist. Then we obtain

Proposition 4. Under $1 < \sigma$

(I) if $0 < \beta < \alpha$, then the optimal path of (*k*, *c*) is the one which converges monotonically to the stationary state $E(\dot{h}, \bar{c})$ in a finite time $t = T < +\infty$ and stays at E thereafter, by taking $c = c_0 e^{\beta t}$, till $t = \bar{T} < +\infty$ with $c_0 = \alpha \bar{h} e^{-\beta T}$ where T is implicitly defined by (23).

(II) If $0 = \beta < \alpha$, then the optimal path of (*k*, *c*) is $c = c_0 = \alpha k_0$ and $k = k_0$, i.e., to stay in the initial state of $(\alpha k_0, k_0)$.

(III) If $\beta < 0 < \alpha$, then the optimal path of (*k*, *c*) is by taking $c_0 = \alpha k_0$, expressed as $c = c_0 e^{\beta t}$ till $t = T$ and $c = \bar{c} = \alpha \bar{h}$ for $t \geq T$, where T is implicitly defined by (23).

That is, after the consumption keeps decreasing but the capital keeps increasing for a while, the economy reaches the full employment equilibrium $E(\bar{k}, \bar{c})$.

(IV) If $\beta < \alpha \leq 0$, then the optimal path of (k, c) converges toward the origin monotonically as time goes to infinity (poverty trap).

Proof. See Appendix V.

The results of Proposition 4 are similar to those of Propositions 1 and 3 in the sense that (I) if $\beta > 0$, then the economy converges to the full employment equilibrium in a finite time, (II) if $\beta = 0$, then the economy stays in the initial state and, (IV) if $\beta < \alpha \leq 0$, then the economy is on the poverty trap. The only difference from Proposition 1 is (III) $\beta < 0 < \alpha$. In case of $\sigma < 1$, Proposition 2 holds for (III). However in case of $\sigma > 1$ irrespective of the initial amount of capital k_0, the consumption keeps decreasing while the capital keeps increasing and the economy reaches to the full employment equilibrium $E(\bar{k}, \bar{c})$ in a finite time T.

Concluding Remarks

Our conclusions are radically different from Dixit (1968)'. He obtained that the stationary state is a globally stable saddle point just like a standard Ramsey model without any possibility of entering into a poverty trap. Furthermore despite of its rather complicated saving rule, the level of technology does not seem to contribute the characteristics of the optimal path.

Next, the comparisons between this labor-surplus economy and the standard full-employment model should be made. As is well known in the standard optimum growth model with full-employment of labor and neoclassical production function, the economy can achieve the unique stationary state from any initial state of per capita capital, while maintaining of course full-employment of labor. However in this labor-surplus economy the achieving such a full-employment stationary is possible if the level of average capital productivity $\bar{p}$ is higher than the discount rate ρ irrespective of the level of initial per capita capital. Furthermore if the elasticity of marginal felicity σ is less than 1, then such a full-employment equilibrium can be also attained by increasing initial per capita capital, say by importing capital in case of $\beta < 0 < \alpha$ under certain condition (A.2).

Third, the stationary value of the average capital productivity $\overline{p}$ is lower in the developing economy not due to the capital scarcity but due to the high and fixed wage rate $\overline{w}$ ($\overline{w}$ is higher if and only if $\overline{p}$ is lower), where the possibility of a flexible wage rate is excluded from the beginning. The above conclusion suggests that the possible way to get out of the optimal, yet so called poverty trap path, and to reach the optimal path converging to the full-employment state, is to increase the average labor productivity $\overline{v} = f(\overline{h})$ by means of a big push (a synchronized expansion in many sectors) or imports of technology. In addition, in case of $\sigma < 1$, another way to reach such a path is to increase initial per capita capital.

Forth, the most crucial assumption of this model is the rigidity of the wage rate. If this assumption is relaxed so that it is adjusted to the level of marginal labor productivity with some time lags, then such a revised model would be more realistic and hence seems worth trying as a next trap.

Appendix I

First we note

$$f(1) = 1/b - a^{\sigma-1}\alpha^{-\sigma}(\beta/\alpha - 1)^{1-\sigma} = 1/b - 1/\alpha > 0$$

from $\alpha = \overline{p} - n > \rho - n = b > 0$.

Next we suppose $f(x) = 0$ holds for contradiction.

Then

$$f(x) = 0 \Leftrightarrow (1/a + 1/b)x - 1/a = a^{\sigma-1}\alpha^{-\sigma}\left(\frac{\beta}{\alpha}x - 1\right)^{1-\sigma}$$

Fig. A.1

As shown in Fig. A.1 two curves $(1/a + 1/b)x - 1/a$ and $a^{\sigma}\alpha^{-\sigma}\left(\frac{\beta}{\alpha}x - 1\right)^{1-\sigma}$ never intersects for $x \geq 1$. In fact, the slope of the latter curve at $x = 1$ is equal to $a^{\sigma-1}\alpha^{-\sigma}(1-\sigma)\beta\alpha^{-1}(\beta/\alpha - 1)^{-\sigma} = (1-\sigma)\beta/a\alpha = (a+b)/a\alpha$, which is less than $1/a + 1/b$, the slope of $(1/a + 1/b)x - 1$ curve.

In fact

$$(a+b)/a\alpha < 1/a + 1/b \Leftrightarrow 1/\alpha + b/a\alpha < 1/a + 1/b$$

$$\Leftrightarrow 1/b - 1/\alpha > b/a\alpha - 1/a \Leftrightarrow 1/b - 1/\alpha > (b/a)(1/\alpha - 1/b) \Leftrightarrow 1 > -b/a$$

from $1/b > 1/\alpha$

This shows $f(x) > 0 \quad {}^{\forall}x \geq 1$.

Appendix II

The value function U is $U = (1-\sigma)^{-1}\left\{c_0^{1-\sigma}(1 - e^{aT})/(\alpha - \beta) + \overline{c}^{1-\sigma}e^{-bT}/b\right\}$.

Since $t = T$ is implicitly defined by (15) as

$$\overline{h} = (k_0 - c_0/(\alpha - \beta))e^{\alpha T} + (c_0/(\alpha - \beta))e^{\beta T}. \quad \text{(A-1)}$$

for given k_0 and $\overline{h}$, c_0 is a function of T such that

$$dc_0/dT = (\alpha\overline{h} - c_0 e^{\beta T})e^{-\alpha T}(\alpha - \beta)/(1 - e^{-(\alpha-\beta)T}) > 0 \quad \text{(A-2)}$$

from $\beta < 0 < \alpha$.

Now let

$$(1-\sigma)U = f(T) = c_0^{1-\sigma}(1 - e^{aT})/(\alpha - \beta) + \overline{c}^{1-\sigma}e^{-bT}/b.$$

Then

$$f'(T) = (1-\sigma)c_0^{-\sigma}c_0'(1-e^{aT})/(\alpha-\beta) - c_0^{1-\sigma}ae^{aT}/(\alpha-\beta) - \bar{c}^{1-\sigma}e^{-bT}$$
$$= (1-\sigma)c_0^{-\sigma}(\alpha\bar{h} - c_0e^{\beta T})e^{-\alpha T} + c_0^{1-\sigma}e^{aT} - \bar{c}^{1-\sigma}e^{-bT},$$

by substituting (A-2) into $f'(T)$. Then

$$f'(T) = (1-\sigma)c_0^{-\sigma}\alpha\bar{h}e^{-\alpha T} + \sigma c_0^{1-\sigma}e^{aT} - \bar{c}^{1-\sigma}e^{-bT},$$

and hence

$$\begin{aligned} e^{bT}f'(T) &= (1-\sigma)c_0^{-\sigma}\bar{c}e^{bT-\alpha T} + \sigma c_0^{1-\sigma}e^{(1-\sigma)\beta T} - \bar{c}^{1-\sigma} \\ &= (1-\sigma)c_0^{-\sigma}\bar{c}e^{\sigma\beta T} + \sigma c_0^{1-\sigma}e^{(1-\sigma)\beta T} - \bar{c}^{1-\sigma} = h(T) \end{aligned} \quad \text{(A-3)}$$

Here we observe

$$(1-\sigma)c_0^{-\sigma}\bar{c} + \sigma c_0^{1-\sigma} > \bar{c}^{1-\sigma}.$$

In fact let $g(x) = (1-\sigma)x^{-\sigma}\bar{c} + \sigma x^{1-\sigma} - \bar{c}^{1-\sigma}$ where $x = c_0(<\bar{c})$. Clearly $g(\bar{c}) = 0$ and $g'(x) = \sigma(1-\sigma)x^{-\sigma-1}(x-\bar{c}) < 0$ for $x < \bar{c}$. Hence $g(x) > 0$ follows.

Furthermore $h(0) = g(x) > 0$ and $h'(T) = -\sigma(1-\sigma)\beta c_0^{-\sigma}\bar{c}e^{-\sigma\beta T} + \sigma(1-\sigma)\beta c_0^{1-\sigma}e^{(1-\sigma)\beta T} > 0$ from $-\sigma(1-\sigma)\beta > 0$, $c_0^{-\sigma}\bar{c} > c_0^{1-\sigma}$ and $e^{-\sigma\beta T} > 1 > e^{(1-\sigma)\beta T}$ when $\beta < 0$, implying $h(T) > 0$ and hence $f'(T) > 0$. This implies $f(T)$ is maximized by taking $c_0 = \alpha k_0$ from $dc_0/dT > 0$.

Hence we obtain U is expressed as

$$U = (1-\sigma)^{-1}\left\{\alpha^{1-\alpha}k_0^{1-\sigma}(1-e^{aT})/(\alpha-\beta) + (\alpha\bar{h})^{1-\sigma}e^{-bT}/b\right\}.$$

Next by substituting $c_0 = \alpha k_0$ into (A-1), we obtain k_0 as a decreasing function of T[13/] so that

$$k_0 = \bar{h}e^{-\beta T}(\alpha - \beta e^{-aT})^{-1}(\alpha-\beta). \quad \text{(A-4)}$$

Then $\tilde{U}$ and U are expressed respectively as

$$(1-\sigma)\tilde{U} = (\alpha-\beta)^{-\sigma}k_0^{1-\sigma} = (\alpha-\beta)^{1-2\sigma}\bar{h}^{1-\sigma}e^{-(1-\sigma)\beta T}(\alpha-\beta e^{-aT})^{\sigma-1}$$

and

$$(1-\sigma)U = \alpha^{1-\sigma}\bar{h}^{1-\sigma}e^{-(1-\sigma)\beta T}(\alpha-\beta e^{-aT})^{\sigma-1}(\alpha-\beta)^{-\sigma}(1-e^{aT}) + \alpha^{1-\sigma}\bar{h}^{1-\sigma}e^{-bT}/b.$$

Here recalling $(1-\sigma)\beta = \beta - \alpha + \rho - n = a + b$ (21), we can rewrite the above as

$$\hat{G} = e^{bT}(1-\sigma)\tilde{U} = (\alpha-\beta)^{1-2\sigma}\bar{h}^{1-\sigma}e^{-aT}(\alpha-\beta e^{-aT})^{\sigma-1}$$

and

$$\hat{H} = e^{bT}(1-\sigma)U = \alpha^{1-\sigma}\bar{h}^{1-\sigma}e^{-aT}(\alpha - \beta e^{-aT})^{\sigma-1}(\alpha-\beta)^{-\sigma}(1-e^{aT}) + \alpha^{1-\sigma}\bar{h}^{1-\sigma}/b\,.$$

Here at $T = 0$ (i.e., $k_0 = \bar{h}$),

$$G = (\alpha-\beta)^{-\sigma}\bar{h}^{1-\sigma} \tag{A-5}$$

and

$$H = \alpha^{1-\sigma}\bar{h}^{1-\sigma}/b\,. \tag{A-6}$$

Furthermore by letting $e^{-aT} = x$, we rewrite $\hat{G}$ and $\hat{H}$ as

$$G = \hat{G}\bar{h}^{\sigma-1} = G(x) = (\alpha-\beta)^{1-2\sigma}x(\alpha-\beta x)^{\sigma-1}$$

and

$$H = \hat{H}\bar{h}^{\sigma-1} = H(x) = \alpha^{1-\sigma}x(\alpha-\beta x)^{\sigma-1}(\alpha-\beta)^{-\sigma}(1-x^{-1}) + \alpha^{1-\sigma}/b$$

$$= \alpha^{1-\sigma}(\alpha-\beta)^{-\sigma}(\alpha-\beta x)^{\sigma-1}(x-1) + \alpha^{1-\sigma}/b\,.$$

Hence

$$H > G \iff \alpha^{1-\sigma}(\alpha-\beta)^{-\sigma}(\alpha-\beta x)^{\sigma-1}(x-1) + \alpha^{1-\sigma}/b > (\alpha-\beta)^{1-2\sigma}x(\alpha-\beta x)^{\sigma-1}$$

$$\iff \alpha^{1-\sigma}(\alpha-\beta)^{-\sigma}(x-1) + \alpha^{1-\sigma}b^{-1}(\alpha-\beta x)^{1-\sigma} > (\alpha-\beta)^{1-2\sigma}x$$

$$. \iff \alpha^{1-\sigma}b^{-1}(\alpha-\beta x)^{1-\sigma} > \left\{(\alpha-\beta)^{1-2\sigma} - \alpha^{1-\sigma}(\alpha-\beta)^{-\sigma}\right\}x + \alpha^{1-\sigma}(\alpha-\beta)^{-\sigma}$$

Fig. A.2

By letting $\widetilde{H}(x)$ and $\widetilde{G}(x)$ be respectively the left hand side and the right hand side of the above inequality, we obtain

$$\widetilde{H}(1) = \alpha^{1-\sigma}b^{-1}(\alpha-\beta)^{1-\sigma} > (\alpha-\beta)^{1-2\sigma} = \widetilde{G}(1) \iff \alpha((\alpha-\beta)/\alpha)^{\sigma} > b = \rho - n$$

holds from A. 2.

Furthermore as seen from Fig. A.2, noting

$$A = (\alpha-\beta)^{1-2\sigma} - \alpha^{1-\sigma}(\alpha-\beta)^{-\sigma} = (\alpha-\beta)^{-\sigma}\left\{(\alpha-\beta)^{1-\sigma} - \alpha^{1-\sigma}\right\} > 0$$

from $\beta < 0 < \alpha$.

There exist $x^* \geq 1$ such that

$$\widetilde{H}(x) < \widetilde{G}(x) \quad {}^{\forall}x > x^*.$$

Here recalling T is an decreasing function of k_0 and $a < 0$, we obtain there exists

$k^* < \bar{h}$ such that

$$U < \tilde{U} \quad {}^{\forall}k < k^*$$

where x^* is defined implicitly to be $\tilde{H}(x^*) = \tilde{G}(x^*)$. ■

Appendix III

Since $c = c_0 e^{\beta t}$ holds form (11), for $\beta < 0$ noting $a = \beta - \alpha = n - \rho = -b < 0$ for $\sigma = 1$, $\beta < \alpha$ is seen to be the only possible combination of α and β. Then its value function is expressed as

$$\tilde{U} = \int_0^{\bar{t}} u(c) e^{-bt} dt = \int_0^{\bar{t}} (\log c_0 + \beta t) e^{-bt}$$

$$= (1 - e^{-b\bar{t}}) b^{-1} \log c_0 + \beta \left\{ \left[-(t/b) e^{-bt} - (1/b^2) e^{-bt} \right]_0^{\bar{t}} \right\}$$

$$= (1 - e^{-b\bar{t}}) b^{-1} \log c_0 - (\beta / b) \left\{ e^{-b\bar{t}} (\bar{t} + 1/b) - 1/b \right\}$$

$$= (k_0 / c_0) \log c_0 - (\beta / b) \left[-(1 - bk_0 / c_0) b^{-1} \left\{ \log(1 - bk_0 / c_0) - 1 \right\} - 1/b \right].$$

Here $e^{a\bar{t}} = e^{-b\bar{t}} = 1 - bk_0 / c_0$ from (22), and $\int_0^{\bar{t}} t e^{-bt} dt = \left[-(t/b) e^{-bt} - (1/b^2) e^{-bt} \right]_0^{\bar{t}}$ are made use of.

Appendix IV

By calculation we obtain for $\beta < 0 < \alpha$ with $\sigma = 1$,

$$U = (\log c_0)(1 - e^{-bT}) / b - (\beta / b) \left\{ e^{-bT} (T + 1/b) - 1/b \right\} + (\log \bar{c}) e^{-bT} / b \quad .$$

(See Appendix III for similar calculation.)

Next we note (34) defines implicitly c_0 as a function of T just as (A-2) so that

$$c_0' = dc_0 / dT = (\alpha \bar{h} - c_0 e^{\beta T}) e^{-\alpha T} (\alpha - \beta) / (1 - e^{aT}) > 0 \qquad \text{(A-2)}$$

from $c_0 < \alpha \bar{h} = \bar{c}$ and $\beta < 0$. Recalling this, we obtain by totally differentiating

$f(T)=U$ with respect to *T*,

$$f'(T)=(1/c_0)(1-e^{-bT})b^{-1}c_0'+(\log c_0)e^{-bT}-(\log\bar{c})e^{-bT}-(\beta/b)\left\{-be^{-bT}(T+1/b)+e^{-bT}\right\}$$

$$=(1/c_0)(1-e^{-bT})b^{-1}(\alpha\bar{h}-c_0e^{\beta T})(1-e^{aT})^{-1}e^{-\alpha T}(\alpha-\beta)+(\log c_0)e^{-bT}-(\log\bar{c})e^{-bT}+\beta e^{-bT}\cdot T$$

(by substituting c_0')

$$=(\alpha\bar{h}-c_0e^{\beta T})c_0^{-1}e^{aT}+(\log c_0)e^{-bT}-(\log\bar{c})e^{-bT}+\beta e^{-bT}\cdot T \quad \text{from} \quad a=-b \quad \text{for} \quad \sigma=1 .$$

Then we observe

$$e^{bT}f'(T)=(\alpha\bar{h}e^{-\beta T}-c_0)c_0^{-1}e^{2\alpha T}+\log c_0-\log\bar{c}+\beta T \text{ (from } b-\alpha=-a-\beta=-\beta\text{)}$$

$$>\left\{\alpha\bar{h}(1-\beta T)-c_0\right\}c_0^{-1}+\log c_0-\log\bar{c}+\beta T \text{ (from } e^{-\beta T}>1-\beta T \text{ and } e^{2\alpha T}>1\text{)}$$

$$>(\alpha\bar{h}-c_0)c_0^{-1}+\log c_0-\log\bar{c}>0 \quad \text{from} \quad -\beta T(\bar{c}/c_0-1)>0 .$$

In fact by letting

$$g(x)=(\bar{c}-x)/x+\log x-\log\bar{c}$$

with $x=c_0$, we obtain

$$g(\bar{c})=0 \quad \text{where} \quad x=c_0<\alpha\bar{h}=\bar{c}$$

and

$$g'(x)=-\bar{c}/x^2+1/x=(x-\bar{c})x^{-2}<0$$

we obtain $g(x)>0$. Hence $f(T)$ increases as *T* (and hence c_0) increases, implying that $f(T)$ is maximized by taking $c_0=\alpha k_0$ since $c_0\le\alpha k_0$, is assumed for $\bar{U}$.

Then *U* is expressed as

$$U=(\log\alpha k_0)(1-e^{-bT})/b-(\beta/b)\left\{e^{-bT}(1+1/b)-1/b\right\}+(\log\alpha\bar{h})e^{-bT}/b .$$

Here by substituting

$$k_0=\bar{h}(\alpha-\beta)e^{-\beta T}(\alpha-\beta e^{-aT})^{-1}$$

obtained from (34) into *U* and $\tilde{U}$, we observe

$$H=b\cdot U=\left\{\log\alpha+\log\bar{h}-\beta T-\log(\alpha-\beta e^{-aT})+\log(\alpha-\beta)\right\}(1-e^{-bT})$$
$$-\beta\left\{e^{-bT}(T+1/b)-1/b\right\}+(\log\alpha+\log\bar{h})e^{-bT}$$

and

$$G=b\cdot\tilde{U}=\log b+\log\bar{h}-\beta T-\log(\alpha-\beta e^{-aT})+\log(\alpha-\beta)+\beta/b .$$

Then we observe, by calculation

$$H - G = (\log\alpha)(1 - e^{-bT}) - \left\{\log\bar{h} - \beta T - \log(\alpha - \beta e^{-aT}) + \log(\alpha - \beta)\right\} e^{-bT}$$
$$- \beta \cdot e^{-bT}(T + 1/b) + (\log\alpha + \log\bar{h})e^{-bT} - \log b$$
$$= \log\alpha + \left\{\log(\alpha - \beta e^{-aT}) - \log(\alpha - \beta)\right\} e^{-bT} - (\beta / b)e^{-bT} - \log b .$$

At $T = 0$ (i.e., $k_0 = \bar{h}$),

$$H - G = \log\alpha - \log b - \beta / b < 0 \Leftrightarrow -\beta/(\alpha - \beta) < \log(\alpha - \beta)/\alpha ,$$

which is true for any $\beta < 0$ and $\alpha > 0$.

Fig. A.3

In fact, we can draw the graphs of $\log\alpha/(\alpha - \beta)$ and $-\beta(\alpha - \beta)$ in Fig. A. 3, noting both $\tilde{g}(-\beta) = \log(\alpha - \beta)/\alpha$ and $\tilde{h}(-\beta) = -\beta/(\alpha - \beta)$ to be increasing functions of $-\beta$, from $g'(x) = 1/(\alpha + x)$ and $\tilde{h}'(x) = \alpha/(\alpha + x)^2$. Then from $\tilde{h}(0) = \tilde{g}(0) = 0$ and $\tilde{h}' - \tilde{g}' = (\alpha/(\alpha + x) - 1)/(\alpha + x) < 0$ we obtain $\tilde{h} < \tilde{g}$.

Furthermore by letting

$$F(x) = (H - G)x = (\log\alpha - \log(\alpha - \beta))x + \log(\alpha - \beta x) - \log(\alpha - \beta) - \beta / b$$

where $x = e^{bT} \geq 1$ with $b = -a$, we obtain

$$F(1) = \log\alpha - \log(\alpha - \beta) - \beta/(\alpha - \beta) < 0 ,$$

as seen above,

and

$$F'(x) = \log\alpha - \log(\alpha - \beta) - \beta/(\alpha - \beta x) < 0$$

from

$$-\beta/(\alpha - \beta x) \leq -\beta/(\alpha - \beta) < \log(\alpha - \beta)/\alpha .$$

Hence $F(x) < 0$ follows.

This implies $G > H$ holds always.

Appendix V

What is needed to prove is first, for (II) $0=\beta<\alpha$, $c=c_0=\alpha k_0$ and $k=k_0$ is in fact the optimal path, and second for (III) $\beta<0<\alpha$, the optimal choice is $c_0=\alpha k_0$.

For (II), the value function $\widetilde{U}$ obtained from $c=c_0=\alpha k_0$ and $k=k_0$ is

$$\widetilde{U}=(1-\sigma)^{-1}\int_0^\infty c_0^{1-\sigma}e^{-bT}dt=(1-\sigma)^{-1}(\alpha k_0)^{1-\sigma}/b. \tag{A-7}$$

On the other hand by taking $c_0\le\alpha k_0$ and $c=c_0$ till $t=T$ and $c=\bar{c}=\alpha\bar{h}$ for $t\ge T$ the value function U is expressed as

$$U=(1-\sigma)^{-1}\left(\int_0^T c_0^{1-\sigma}e^{-bT}dt+\int_T^\infty \bar{c}^{1-\sigma}e^{-bT}dt\right)=(1-\sigma)^{-1}\left\{c_0^{1-\sigma}(1-e^{-bT})/b+\bar{c}^{1-\sigma}e^{-bT}/b\right\}. \tag{A-8}$$

Just as far as the case of $0<\sigma<1$, we obtain

$$e^{\alpha T}=(\bar{h}-c_0/\alpha)/(k_0-c_0/\alpha) \tag{A-9}$$

from (15) with $\beta=0$ and $k=\bar{h}$, and by substitution. (A-8) is expressed as

$$b(1-\sigma)U=c_0^{1-\sigma}(1-(k_0-c_0/\alpha)/(\bar{h}-c_0/\alpha))+\bar{c}^{1-\sigma}(k_0-c_0/\alpha)/(\bar{h}-c_0/\alpha)=F(c_0)$$

Then $F(\alpha k_0)=(\alpha k_0)^{1-\sigma}>0$ and

$$F'(c_0)=(1-\sigma)c_0^{-\sigma}(1-(k_0-c_0/\alpha)/(\bar{h}-c_0/\alpha))$$

$$-c_0^{1-\sigma}(k_0-\bar{h})/(\bar{h}-c_0/\alpha)^2\alpha+\bar{c}^{1-\sigma}(k_0-\bar{h})/(\bar{h}-c_0/\alpha)^2\alpha$$

$$=(\bar{h}-c_0/\alpha)^{-2}(\bar{h}-k_0)\left\{(1-\sigma)c_0^{-\sigma}(\bar{h}-c_0/\alpha)+c_0^{1-\sigma}/\alpha-\bar{c}^{1-\sigma}/\alpha\right\}.$$

Here the inside of the curly bracket is equal to

$$\sigma c_0^{1-\sigma}/\alpha+(1-\sigma)c_0^{-\sigma}\bar{h}-\bar{c}^{1-\sigma}/\alpha$$

$$=\sigma c_0^{1-\sigma}(c_0/\alpha-\bar{h})+c_0^{-\sigma}h-\bar{c}^{1-\sigma}/\alpha=\widetilde{F}(c_0).$$

Noting $\widetilde{F}(\bar{c})=0$ and $\widetilde{F}'(c_0)=\sigma(1-\sigma)c_0^{-\sigma-1}(c_0/\alpha-\bar{h})\ge\sigma(1-\sigma)c_0^{-\sigma-1}(k_0-\bar{h})>0$, we obtain $F'(c_0)<0$. Hence $F(c_0)$ is minimized (i.e., U is maximized) by taking $c_0=\alpha k_0$, and therefore $T=+\infty$.

Second for (III) $\beta<0<\alpha$, the value function obtained by taking $c_0\le\alpha k_0$ is expressed as,

$$U = (1-\sigma)^{-1}\left(\int_0^T c_0^{1-\sigma} e^{-bT} dt + \int_T^\infty \overline{c}_0^{1-\sigma} e^{-bT} dt\right)$$

$$= (1-\sigma)^{-1}\left\{c_0^{1-\sigma}(1-e^{aT})/(\alpha-\beta) + \overline{c}^{1-\sigma} e^{-bT}/b\right\},$$

as shown in Appendix II. Then we follow the same method as drawn in Appendix II, and obtain from $b-\alpha = -\sigma\beta$,

$$h(T) = e^{bT} f'(T) = (1-\sigma)c_0^{-\sigma}\overline{c}e^{bT-\alpha T} + \sigma c_0^{1-\sigma} e^{(1-\sigma)\beta T} - \overline{c}^{1-\sigma}$$

$$= (1-\sigma)c_0^{-\sigma}\overline{c}e^{-\sigma\beta T} + \sigma c_0^{1-\sigma} e^{(1-\sigma)\beta T} - \overline{c}^{1-\sigma}, \qquad \text{(A-10)}$$

just as (A-3). Furthermore let $g(x) = (1-\sigma)x^{-\sigma}\overline{c} + \sigma c_0^{1-\sigma} - \overline{c}^{1-\sigma}$ where $x = c_0 (< \overline{c})$.

Clearly $g(\overline{c}) = h(0) = 0$ and $g'(x) = -(1-\sigma)\sigma\overline{c}x^{-\sigma-1} + \sigma(1-\sigma)x^{-\sigma}$ $= -(1-\sigma)\sigma x^{-\sigma-1}(\overline{c}-x) > 0$ from $\sigma > 1$ and $\overline{c} > x$, implying $g(x) = h(0) < 0$.

Next we observe $h'(T) = -(1-\sigma)\sigma c_0^{-\sigma}\overline{c}e^{-\sigma\beta T} + \sigma(1-\sigma)\beta c_0^{1-\sigma} e^{(1-\sigma)\beta T}$

$= (\sigma-1)\sigma\beta c_0^{-\sigma}(\overline{c}e^{-\sigma\beta T} - c_0 e^{(1-\sigma)\beta T}) < 0$ from $\sigma > 1$, $\beta < 0$ and $-\sigma\beta > (1-\sigma)\beta$.

Hence $h(T) < 0$, implying $(1-\sigma)bU = f(T)$ is minimized and hence U is maximized by taking $c_0 = \alpha k_0$ from (A-2).

Figures

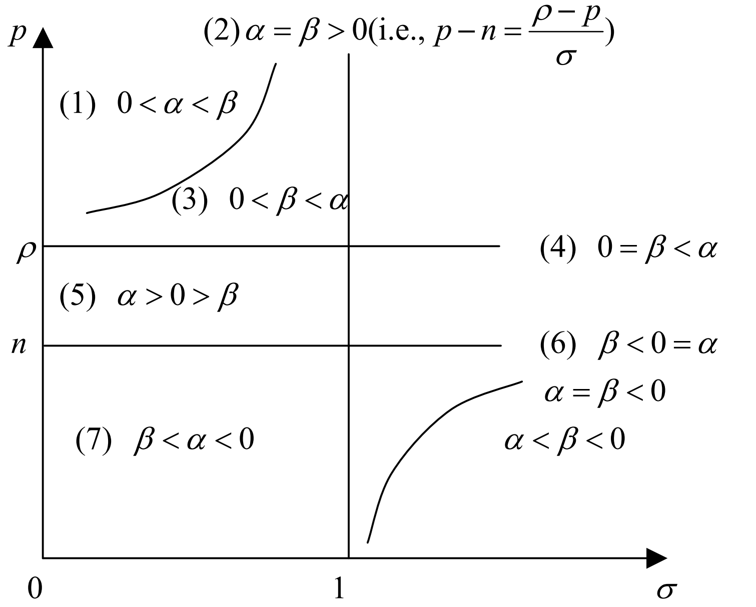


Fig.1 The area of (α, β)

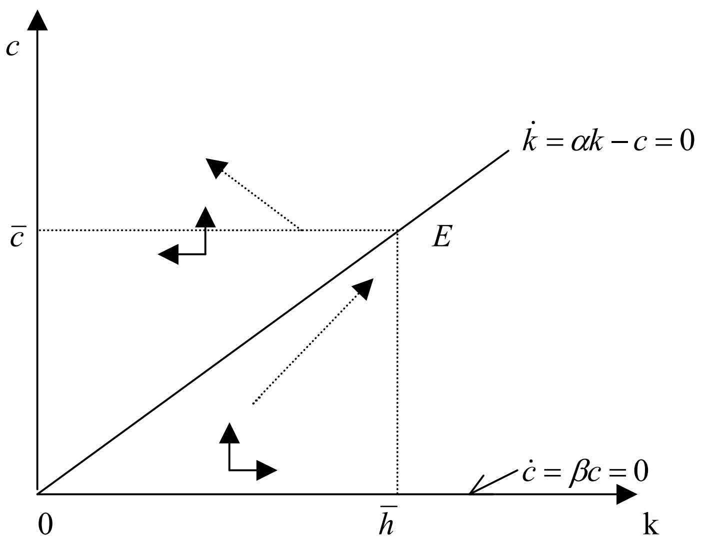


Fig. 2

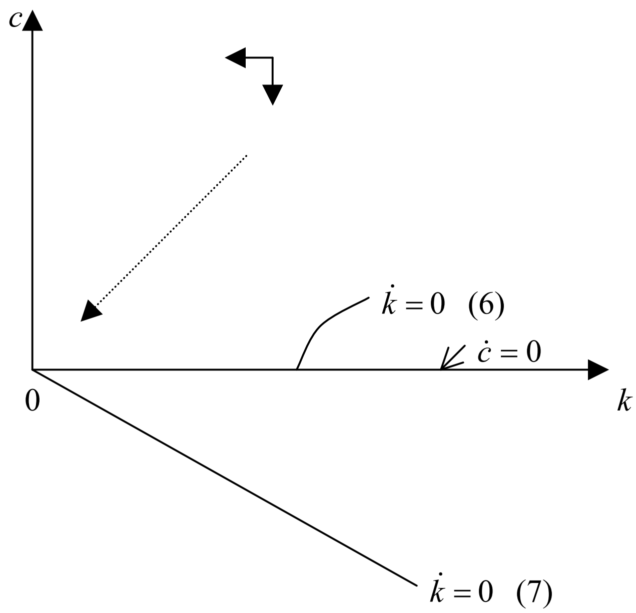


Fig. 3

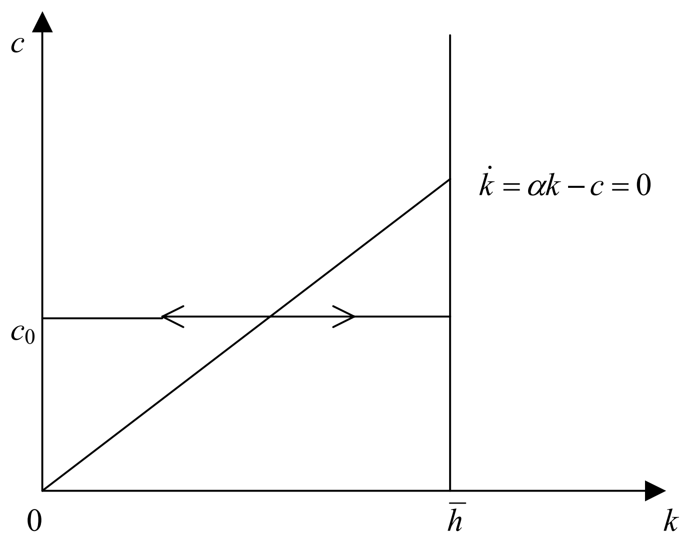
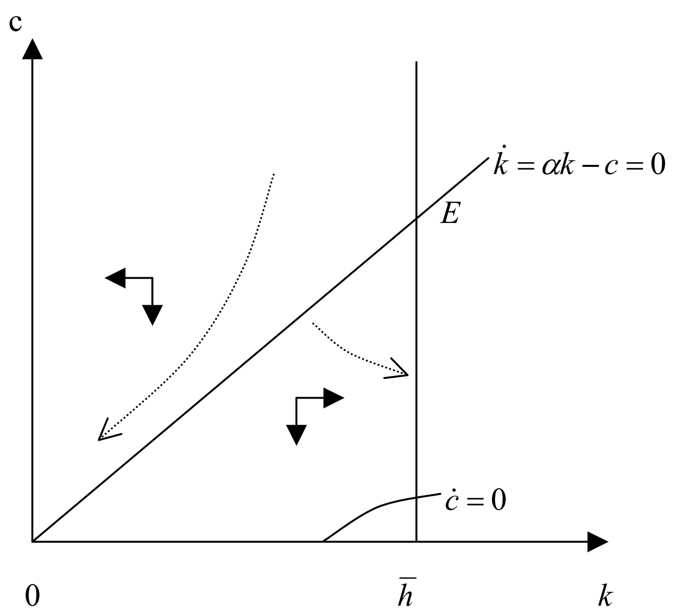
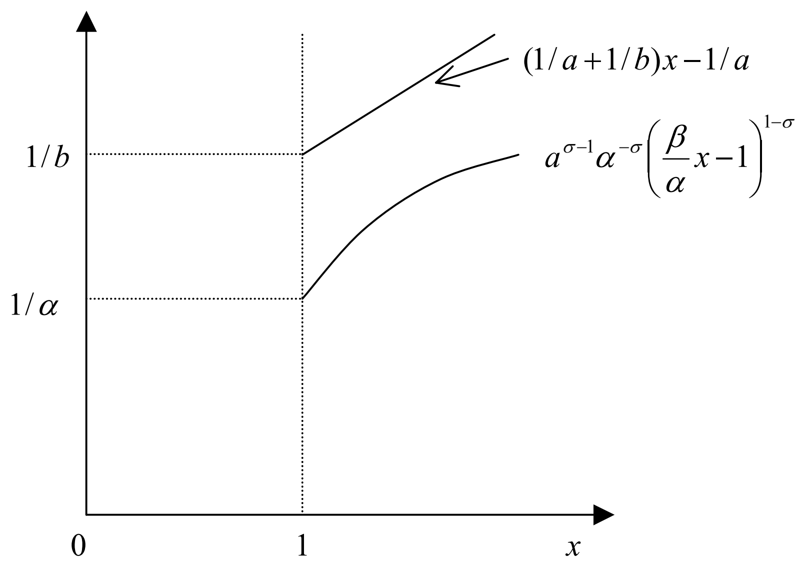


Fig. 4


c
$\dot{k} = \alpha k - c = 0$
E
$\dot{c} = 0$
0
$\bar{h}$
k
Fig.5


$(1/a+1/b)x-1/a$
$a^{\sigma-1}\alpha^{-\sigma}\left(\frac{\beta}{\alpha}x-1\right)^{1-\sigma}$
$1/b$
$1/\alpha$
0
1
x
Fig. A. 1

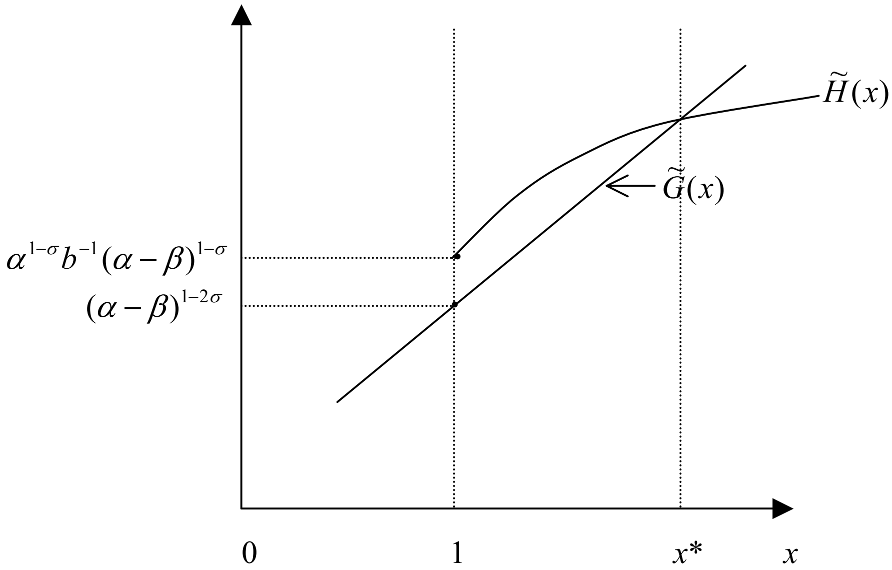

$\widetilde{H}(x)$
$\widetilde{G}(x)$
$\alpha^{1-\sigma}b^{-1}(\alpha-\beta)^{1-\sigma}$
$(\alpha-\beta)^{1-2\sigma}$
0
1
x^*
x

Fig. A. 2

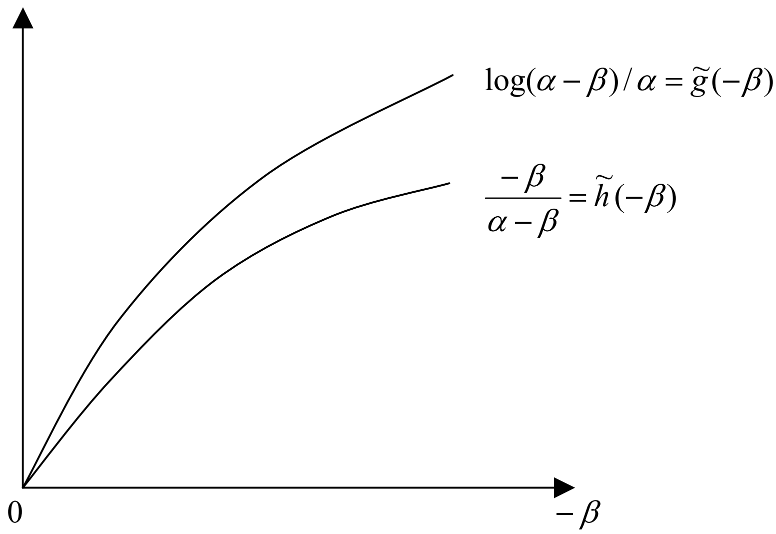

$\log(\alpha-\beta)/\alpha=\widetilde{g}(-\beta)$
$\dfrac{-\beta}{\alpha-\beta}=\widetilde{h}(-\beta)$
0
$-\beta$

Fig. A. 3

Footnotes

1. For a further lucid survey of vicious circle of poverty, see, e.g., Basu (1997) Chapter 2.
2. To increase the aggregate productivity by means of balanced growth was first advocated by Rosenstein-Rodan (1943) and then Nurkse (1953). Lately Murphy, Shleifer and Vishny (1989) rigorously formulated the formers' idea.
3. Majumdar and Mitra (1995) formulated a model where a country cannot escape from the poverty trap as far as it stays in autarky, but can get out of this if it opens up to trade.
4. Let $k = (a + bt)e^{\alpha t}$ with a and b being constant. Then $k_0 = a$ and from $\dot{k} = (a\alpha + b\alpha t + b)e^{\alpha t}$ and (6) with $t = 0$, we obtain $a\alpha + b = k_0\bar{p} - c_0 - nk$. Here recalling $\alpha = \bar{p} - n$, we observe $b = -c_0$.
5. If both k and c become zero in a finite time $\bar{t} < +\infty$, then the Hamiltonian H must be zero at $t = \bar{t}$ as a transversality condition. From the transversality conditions for free finite terminal problem of optimal control theory. See. e.g., Léonard and Long (1992), Chapter 7.
6. Let $f(T) = \alpha(T + 1/b)$ and $g(T) = (\alpha T + 1)^{1-\sigma}$. Then from $f(0) = \alpha/b > 1 = g(0)$ and $g'(0) = (1-\sigma)\alpha < \alpha/b = f'(0)$, noting $g(T)$ is concave we observe that $f(T) > g(T)$ follows.
7. Just as for the case (1), we note $U > \widetilde{U} \Leftrightarrow (x-1)/(\alpha - \beta) + 1/b > (\alpha-\beta)^{\sigma-1}\alpha^{-\sigma}x(1 - \frac{\beta}{\alpha}x^{-1})^{1-\sigma} = (\alpha-\beta)^{\sigma-1}\alpha^{-\sigma}x^{\sigma}(x - \frac{\beta}{\alpha})^{1-\sigma}$ $\forall x \geq 1$ by letting $e^{-aT} = x$, recalling $a = \beta - \alpha < 0$ for case (3). By letting the left hand side and the right hand side $f(x)$ and $g(x)$ respectively we obtain $f(1) = 1/b$ and $g(1) = 1/\alpha$ and hence $f(1) > g(1)$. Here we note $g(x)$ to be concave. In fact $g'(x) = (\alpha-\beta)^{\sigma-1}\alpha^{-\sigma}\left\{\sigma x^{\sigma-1}(x - \frac{\beta}{\alpha})^{1-\sigma} + (1-\sigma)x^{\sigma}(x - \frac{\beta}{\alpha})^{-\sigma}\right\}$ and $g''(x) = -(\alpha-\beta)^{\sigma-1}\alpha^{-\sigma}\cdot(1-\sigma)\sigma x^{\sigma-2}(x - \frac{\beta}{\alpha})^{-\sigma-1}(\alpha^2/\beta^2) < 0$. Furthermore since $f'(1) = 1/(\alpha-\beta)$ and $g'(1) = \sigma/\alpha + (1-\sigma)/(\alpha-\beta)$, we observe $f'(1) - g'(1) = \sigma\{1/(\alpha-\beta) - 1/\alpha\} > 0$ and hence $f(x) > g(x)$ holds $\forall x \geq 1$.

8. Let the right hand side of (32) be $f(x) = x^{1-\sigma}(1-(k_0 - x/\alpha)/(\overline{h} - x/\alpha)) + \overline{c}^{1-\sigma}(k_0 - x/\alpha)/(\overline{h} - x/\alpha)$ where $x = c_0 < \alpha k_0 < \alpha\overline{h} = \overline{c}$. Then from $f'(x) = (\overline{h} - k_0)\{(1-\sigma)x^{-\sigma}/(\overline{h} - x/\alpha) + x^{1-\sigma}/(\overline{h} - x/\alpha)^2\alpha - \overline{c}^{1-\sigma}/\alpha((\overline{h} - x/\alpha)^2\}$, we observe $\operatorname{sgn} f'(x) = \operatorname{sgn}\{(1-\sigma)x^{-\sigma}(\overline{h} - x/\alpha)\alpha + x^{1-\sigma} - \overline{c}^{1-\sigma}\}$ $\operatorname{sgn}\{(1-\sigma)x^{-\sigma}\overline{h}\alpha + \sigma x^{1-\sigma} - \overline{c}^{1-\sigma}\}$. Let the inside of the curly bracket be $g(x)$. Then $g'(x) = -\sigma(1-\sigma)x^{-\sigma-1}\overline{c} + \sigma(1-\sigma)x^{-\sigma} = -\sigma(1-\sigma)x^{-\sigma-1}(\overline{c} - x) < 0$, and $g(\overline{c}) = 0$ imply $g(x) > 0$ for $x \in [0, \alpha k_0)$. This shows $f'(x) > 0$ for $x \in [0, \alpha k_0)$.
9. A. 2 holds if σ is near 1. In fact if $\sigma = 1$, then the both sides of the inequality is equal. Let $f(\sigma) = \log \alpha^{1-\sigma}(\alpha - \beta)^{\sigma}$. Then $f'(\sigma) = -\log\alpha + \log(\alpha - \beta) + \sigma(\alpha - \beta)^{-1}\beta/\alpha$, and hence $f'(1) = -\log(\bar{p} - n) + \log(\rho - n) + (\bar{p} - \rho)/(\rho - n) < 0$ when $\rho - n < 1$ and $\bar{p} < \rho$ implying $f(\sigma) > \log(\rho - n)$ if σ is near 1.
10. See Appendix III for its derivation.
11. $$\lim_{x\to 0} x\log x = \frac{\lim_{x\to 0}\log x}{\lim_{x\to 0}\frac{1}{x}} = \frac{\lim_{x\to 0}\frac{1}{x}}{\lim_{x\to 0} -\frac{1}{x^2}} = \lim_{x\to 0} x = 0$$ from L'hopital's rule.
12. By letting $x = c_0$ and $g(x) = (\overline{c} - x)/x + \log x - \log\overline{c}$, we see $g(\overline{c}) = 0$ and $g'(x) < 0$ implying $g(x) > 0$.
13. By calculation we obtain $dk_0/dT = \alpha\beta(\alpha - \beta)^{-1}k_0^2\overline{h}^{-1}(e^{\alpha T} - e^{\beta T}) < 0$ from $\beta < 0$.

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