

# Inequalities and Patience in Catching up

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## Abstract

This paper examines how impatience interacts with inequalities in economic development. We consider two distinct groups of households with intrinsic inequality (e.g., capitalists and workers), and show that (i) under *decreasing* marginal impatience (DMI), an unequal society may be preferable for poor households than communism when every household owns an equal share of asset; (ii) poor households tend to benefit more from positive shocks under DMI than CMI (*constant* marginal impatience). (iii) inequality exhibits a sharp inverted-U shape as more people become rich, which should be good news for developing countries in catching up; (iv) a tax on capital income reduces poor households' income when the fraction of the rich is sufficiently small; (v) immigration into rich countries raises their average income but widens the income gap.

Keywords: China; India; Inequality; Catching up; Marginal Impatience

## 1 Introduction

The recent bestseller by Piketty (2014, English edition) has revived wide interest in the relationship between social inequality and capital-asset ownership. Piketty argues that the world

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today is returning towards "patrimonial capitalism", in which much of the economy is dominated by inherited wealth: Their power is increasing, creating an oligarchy. He thus proposes a global system of progressive wealth taxes, to avoid the vast majority of wealth coming under the control of a tiny minority.

In the real world, we are also witnessing Brexit in Europe and an anti-globalization billionaire becoming the U.S. president, who threatened to build a wall along the U.S.-Mexican borders, cancelled Trans-Pacific Partnership in his first day of office and ordered to disallow the citizens of 7 islamic countries to enter the U.S. It is widely believed that such events were triggered by polarization in the rich countries, stemming from increasing globalization especially large-scale immigration and offshoring.

However, is inequality necessarily bad for the poor. In the present paper, we offer an alternative explanation. As assets are accumulated and reinvested, diminishing returns kick in on the one hand and the demand for labor increases on the other hand, raising the wage rate. These effects are especially strong when we incorporate decreasing marginal impatience (DMI), under which households save and invest more when they become richer, thereby increasing capital accumulation and eventually generating a trickle-down effect to the poor in the long run. This mechanism increases the capital stock, the productivity of workers and the welfare of all households including the poor when the rich becomes richer. Hence, in contrast to the alarm caused by Piketty, we find that inequality may not be so bad after all, on the contrary, it might just be a "growing pain" or even a "necessary evil" on a country's catching-up path, in order to increase the incentives for investment and eventually enlarge the total pie.

Our model is motivated by the experiences of many developing countries, in whose early years of economic development, widespread subsidies are provided to the rich—those fortunate enough to be business owners, such as the policies applied in Special Economic Zones (SEZ)

where tax holidays, export, import and land subsidies are common, hoping to enlarge the pie for the whole country. Some of these countries have achieved great success with such policies and become the so-called "fast growing economies". Yet, their income inequality has also been rising rapidly. For instance, the Chinese Gini coefficient remains well above the warning level of 0.4 set by the United Nations, peaking at 0.55 in 2002 (see Li, Wei and Jing (2005)).<sup>1</sup> China Daily (May 23, 2012) reports that the most affluent 10 percent of the population makes 23 times more than the poorest 10 percent. Further, among the so-called fast growing BRICS countries (Brazil, Russia, India, China and South Africa), the Gini coefficient in South Africa was 0.67 in 2008, followed by Brazil (0.51 in 2012), China (0.47 in 2012) and Russia (0.41 in 2011).<sup>2</sup>

Given the above stylized facts, one naturally asks the following question: could inequality be responsible for the high growth rates in these economies? Indeed, when China started its open-door policy, the then-leader, Deng Xiaoping, in particular stressed to "allow a small fraction of the population to get rich first."<sup>3</sup> Recent studies by Chang, Gu and Tam (2015) and Gu, Li and Tam (2015) find that income inequality is a significant contributor to China's savings glut, which enables the recent Chinese growth that is heavily dependent on investment. Earlier, Banerjee and Duflo (2003) find that with cross-country data, changes in inequality in any direction are associated with reduced growth in the next period.

In the present model, we show that under DMI, the welfare of an economy where everyone owns an equal share of capital (aka communism) is lower than if only some people own capital (aka capitalism). More surprisingly and perhaps politically incorrect, when the fraction of rich households is sufficiently small, subsidizing the rich and taxing the poor raises poor households' welfare in the steady state. The logic is, the lower the share of capitalists in the economy, the

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<sup>1</sup>Even official figures showed the Gini to be 0.491 in 2008 (National Bureau of Statics of China).

<sup>2</sup>See Business Standard News, August 10, 2013.

<sup>3</sup>Deng Xiaoping made such remarks when meeting with a group of U.S. visitors in Setempler 1986.

more they save and the more capital the country accumulates under DMI. Hence, a country with higher inequality accumulates higher capital stock and enjoys higher levels of welfare per capita, *ceteris paribus*. On the contrary, if the same amount of capital stock is spread over more owners, each capitalist saves less and the steady-state welfare becomes lower. As such, the poor may accept inequality to a certain extent, as long as their absolute income rises.

Even better news is, we find that inequality exhibits an inverted-U shape under DMI, i.e., the Gini-coefficient first rises then falls as the share of population owning assets increases, exactly because DMI renders the interest rate to fall in the long run while raising the wage rate.

In the academic literature, quite a number of empirical studies find strong evidence that households are heterogeneous in terms of impatience (e.g., they discount the future at different rates), and this heterogeneity is an important factor in explaining household income inequality, see for instance, Hausman (1979), Becker and Mulligan (1997), Samwick (1997), and Barsky et al. (1997). Lawrance (1991) and Warner and Pleeter (2001) find that more-educated households tend to have lower discount rates than less-educated ones, thus heterogeneous time preference may lead to income and wealth inequality through long-term investment and human capital accumulation. In fact, some studies have found that the marginal propensity to save is considerably higher among wealthier people.<sup>4</sup> As for the microeconomic foundations of DMI, it may arise for the following reasons: The rich may invest more on health, beauty and education, enabling them to live longer and healthier, making them more opportunistic for the future.

Based on the above empirical evidence and stylized facts, we thus incorporate DMI into the present model. In particular, we consider a society without “equal opportunity” to begin with, as is a fact in many developing countries with strong traditional systems (e.g., some Latin

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<sup>4</sup>Frederick et al. (2002) provides a detailed review of this literature.

American countries, China and India, etc.), especially before their reform and take-off periods.<sup>5</sup> To be specific, there exist two types of households that are symmetric in all aspects except that one type owns asset (e.g., capitalists), while the other type (e.g., workers) is unable to own asset and hence consume all income at each point in time. In such economies, the fraction of rich households is small and the financial market is inefficient so poor households can hardly save or borrow.

We rigorously examine how such inequality evolves under globalization, in particular facing technology improvements (such as the telecommunication and Internet revolution in the early 1990s) and international labor migration. In our model, *poor* households tend to benefit more under DMI than CMI, because DMI generates a long-run trickle down effect that is absent under CMI. We find that a positive productivity shock always raises the income gap under CMI (constant marginal impatience), because rich households benefit in more ways or more directly from such shocks while poor households only benefit through changes in the wage rate. This is consistent with standard models such as Acemoglu (2002), who studies the impacts of skilled labor-biased technology improvement and finds it to be a major cause for income inequality in the twenty's century. However, under DMI, an increase in productivity reduces the Gini-coefficient due to the trickle down effect stemming from the rich saving more, and the level income-gap may also fall. These are exactly opposite to Acemoglu (2002).

We can clearly compare the welfare levels of communism (i.e., every household owns an equal share of asset) with CMI and DMI. With CMI, it is lower than the rich's welfare but

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<sup>5</sup>For instance, although slowly being relaxed, the infamous family registration (*hukou*) system in China determines whether one is a peasant, urban worker, or cadre, etc. at birth (!) following the mother. And India is wellknown for the Caste system, which has largely broken down in cities, but persists in rural areas where 72% of India's population resides. Several Latin American countries have the highest Gini index in the world, and such inequalities have been carried over from generation to generation, some even from the colonial period (Ferreira et al., 2004). As such the top 10% population is estimated to own 50% of the total income while the bottom 10% owns only 1.5%, compared to 30% and 2.5% for corresponding groups in developed countries.

higher than the poor's welfare under capitalism (i.e., with household inequality); in contrast, with DMI, it is even lower than the poor's welfare under capitalism. That is to say, with DMI, if society is egalitarian such that every household owns an equal share of asset (i.e., communism), the steady-state welfare is lower than the poor's welfare in a capitalist steady-state. Hence, it may seem that we have assumed asymmetry of households in basic model, but given DMI, inequality of households turns out to be a natural consequence of endogenous time preference, and communism would not be preferred by any group. It is socially inefficient to eradicate inequalities if they are not out of reasonable bounds. And even if both types of households own asset to begin with, the economy will not converge to the steady state where all households have some positive level of asset (i.e., everyone becoming a capitalist), as long as their initial asset holdings differ.<sup>6</sup>

These scenarios may justify policies such as a tax on capital earnings to redistribute income from the rich to the poor, to keep inequality within boundaries, otherwise social instability may arise.<sup>7</sup> The government could also increase expenditure on education such as scholarships and loans for low income families, and subsidies and loans for young entrepreneurs, etc. Nevertheless, the tax has an indirect, negative impact on poor households, especially in economies with severe inequality where capital accumulation may be reduced more. Indeed, we find that the effect of the tax on poor households' income can be reversed as the fraction of rich people rises: it reduces (raises) poor households' income when the fraction is sufficiently low (high).

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<sup>6</sup>In contrast, under IMI the economy converges to the steady state where all households have the same level of asset, even when their initial asset holdings differ. See Appendix 2 in Iwasa and Zhao (2013), and also Epstein (1987).

<sup>7</sup>In China, on the surface, it seems to be GDP growth at all costs: the sudden surge in inequality puts pressure on everybody, to try to become richer, as soon as possible, resulting in landslides of public morality. Many newly constructed roads, bridges, railways, buildings and even food products are of poor quality, causing fatal accidents; Air and water pollution has soared to hazardous levels. These phenomena have generated heated debates in the media, among policy makers and researchers alike. There are soul searching cries in the popular press that China in its rush to modernity should slow down the pace, in order to reduce ever-increasing pollution, to decrease man-made errors and potential disasters, and to make more efficient use of its depleting resources.

As might be the case of present-day China, the fraction of the rich has exceeded a certain level, especially in big cities and along the coast, and it might be time to impose a wealth tax.

We also extend the model to examine immigration into rich countries, a typical phenomenon in this age of globalization. We find that while it raises the average income in these countries, it widens their income gaps. These results match the polarization experiences in many developed countries, in particular, prompting Brexit in the EU and Donald Trump to the U.S. presidency.

In the theoretical literature, Krusell and Smith (1998) demonstrate that introducing time preference heterogeneity can significantly improve the Aiyagari (1994) model in explaining income inequality, and Hendricks (2007) incorporates preference heterogeneity into the life-cycle model of Huggett (1996) to account for wealth inequality. Also, Epstein (1987), Das (2003), Hirose and Ikeda (2008) and Chang (2009) investigate equilibrium stability and uniqueness issues under DMI. In an economy with initial inequality, Ghigliano and Sorger (2002) show that redistribution of wealth may drive the economy from a steady state with strictly positive output to a poverty trap in which output converges asymptotically to zero. In Hirose and Ikeda (2012), the Harberger-Laursen-Metzler effect is examined, where at least one country has IMI in order to obtain saddle-point stability. Benhabib, Bisin and Zhu (2011) demonstrate in an overlapping generations model with under inter-generational transmission of wealth, wealth distribution is a Pareto distribution in the right tail, driven by capital income risk rather than labor income. Subsequently, Benhabib, Bisin and Zhu (2012) show that redistributive fiscal policy with idiosyncratic investment risk and uncertain lifetimes can generate a double Pareto wealth distribution. Different from the above, Uzawa (1968) and Kamihigashi (2000) examine cases of IMI rather than DMI.

In contrast, the present paper focuses on the effects of DMI preference, patience and total factor productivity. The interactions of the two types of households generate interesting results

that are novel in the literature.

## 2 The Basic Model

Under endogenous time preference with DMI, households become more patient when they are richer; in other words, a poorer household consumes a higher fraction of its income than a richer household.

In this paper, we focus on an economy with two types of households, which are symmetric in all aspects except that one type owns asset (i.e., capitalists), while the other type is unable to own asset for some reason (i.e., workers).<sup>8</sup> That is, there exists intrinsic social inequality to begin with, as in many developing countries with strong traditional institutions and customs. In such economies, the fraction of rich households is small and the financial market is inefficient so poor households can hardly save or borrow.

Later we shall extend the basic model to examine cases of all households owning an equal share of asset (i.e., communism). For cases of capitalism where some households own less but positive assets than others, see Epstein (1987), who shows that if there is some wealth gap among households initially, the economy will not reach the steady state where all households have positive levels of assets, because the rich will become richer through asset re-investment.

Our goal is to examine the relationship between inequality and economic growth and how social inequality evolves in transitional especially the fast-growing economies such as the so-called BRICS countries (Brazil, Russia, India, China, and South Africa). In the process of catching up, such economies inevitably face technology upgrading, labor migration, demand

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<sup>8</sup>In Appendix 2 in Iwasa and Zhao (2013), we relax this assumption and show that (i) even if all households are allowed to own assets with different initial asset holdings, under DMI the economy will not converge to the steady state where everybody owns capital; (ii) the steady state we shall examine is the same as the one when the borrowing constraint is binding only for one type of households.



and other shocks. We analyze these issues and government policies including tax reform that might be used to mitigate the existing and possibly widening inequality.

Consider one good that is consumed and saved as capital, whose output is given by

$$Y = F(K, L),$$

where  $K$  and  $L$  are respectively capital stock and labor supply. Production exhibits constant returns to scale technology,

$$k \equiv \frac{K}{L} \quad \text{and} \quad f(k) \equiv F\left(\frac{K}{L}, 1\right).$$

Then, the capital rental rate  $R$  and the wage rate  $w$  are respectively

$$R = f'(k) \quad \text{and} \quad w = f(k) - kf'(k).$$

The household's inelastic labor supply is normalized to one, so that the number of households is denoted by  $L$ , and  $k$  is the capital stock per household or simply capital stock.

## 2.1 The Capital Market and Households' Distinct Incomes

In the basic model, poor households consume all their current income so they do not participate in the capital market. Since their income comes from wages only,  $i^*(= w)$ , it increases in capital accumulation.

In contrast, rich households save a portion of their income as assets. The income for rich households with asset  $a$  is given by  $w + ra$ , while that for poor households (without asset) is only  $w$ , where  $r = R - \delta$  is the interest rate and  $\delta$  is the depreciation rate on capital.

Let  $\theta \in (0, 1]$  denote the fraction of households owning asset in the whole economy. Then  $k = a\theta$  from the market clearing condition for asset,  $a\theta L = K$ .<sup>9</sup> It follows that the income of rich households,  $i$ , becomes a function of  $k$  and  $\theta$  as follows:

$$i(k, \theta) \equiv w(k) + \frac{r(k)k}{\theta},$$

where  $w(k) \equiv f(k) - kf'(k)$  and  $r(k) \equiv f'(k) - \delta$ .

Notice that  $w'(k) = -kf''(k) > 0$ , and  $r'(k) = f''(k) < 0$ ;<sup>10</sup> that is, when the capital stock rises, the wage rate increases but the interest rate decreases. In standard models with (one type of) representative households ( $\theta = 1$ ), the two effects are exactly cancelled out ( $w'(k) + r'(k)k = 0$ ), and hence household income,  $f(k) - \delta k$ , always increases with capital accumulation for any positive interest rate, and a representative household's income is maximized at the *golden rule* level of capital stock where  $R = \delta$ , which yields  $r = 0$ .

In contrast, under  $\theta < 1$  as in the present model, we have two types of households, and these two opposite effects do *not* cancel out. In fact we have

**Lemma 1** *If  $\theta < 1$ , capital accumulation can reduce the income of households holding asset, even when the interest rate is positive:*

$$\begin{aligned} \frac{\partial i(k, \theta)}{\partial k} &= \frac{1}{\theta} [(\theta - 1)w'(k) + r(k)] \\ &= \frac{1}{\theta} [(1 - \theta)kf''(k) + r(k)]. \end{aligned}$$

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<sup>9</sup>Poor households consume all income at each point in time, and hence consumption is  $c^* = w$ . Then, the goods market clears when:

$$(c + \dot{a})\theta L + c^*(1 - \theta)L + \delta K = wL + RK.$$

<sup>10</sup>In what follows, we assume  $w''(k) = -f''(k) - kf'''(k) < 0$  as in the Cobb-Douglas technology.

Some explanations are in order. In the present model with  $\theta < 1$ , some households have no assets, then capital accumulation may reduce the income of rich households if the interest rate is sufficiently small (albeit positive), which occurs if the capital stock  $k$  is large. Such a scenario arises easily if the fraction of rich households is small. The intuition is, from the definition of  $i(k, \theta)$  above, when  $\theta$  is small,  $\frac{r(k)k}{\theta}$  becomes large and rich households' income largely consists of asset income, which declines when the interest rate falls due to capital accumulation.

Indeed, we can obtain

**Lemma 2** *With  $\theta < 1$ , the income of rich households is maximized at a certain capital stock lower than the level given by the golden rule*

Note that in standard models like the Ramsey model, households will accumulate capital to a level lower than that given by the golden rule, and there is no case for capital accumulation to lower households' income along the optimal path to the steady state. However, in the present model with  $\theta < 1$ , it could happen, since capital accumulation lowers the interest rate and their asset income. This in turn stems from the fact that poor households consume all their income, as such, the rich must save an extra portion for the poor.

## 2.2 The Optimization Problem for Rich Households

So far all results are obtained without considering DMI. Next we formally incorporate DMI: each of the rich households maximizes the discounted sum of utility

$$\int_0^{\infty} u(c)X dt, \tag{1a}$$

subject to<sup>11</sup>

$$\dot{a} = w + ra - c, \tag{1b}$$

$$\dot{X} = -\rho(c)X, \tag{1c}$$

where for  $\forall c > 0$ ,  $u(c) > 0$ ;  $u'(c) > 0$ ;  $u''(c) < 0$ . Also,  $X \equiv \exp[-\int_0^t \rho(c)ds]$  is the discount factor at time  $t$  which depends on the past and present levels of consumption through the function  $\rho(\cdot)$ .

Following Das (2003) and Chang (2009), we assume that household preference exhibits DMI as follows,<sup>12</sup>

$$\rho'(c) < 0 < \rho''(c) \text{ for } \forall c > 0, \rho(0) < \infty, \text{ and } \lim_{c \rightarrow \infty} \rho(c) = 0. \tag{2}$$

Intuitively, it says households with a higher income discount future less, since they can afford to defer consumption of additional income and wealth. As mentioned in the Introduction, this assumption is supported by a number of empirical studies, such as Lawrance (1991), Barsky et al. (1997) and Samwick (1997), etc. We are interested in how the degree of impatience affects the evolvement of inequalities in the process of catching up and modernization, especially for developing countries.

Further, expression (1c) implies the rate at which  $X$  decreases is  $\rho(c)$ . To be more precise, with an increase in consumption  $c$ , expression  $X$  decreases at a constant speed under CMI, but at a decreasing speed under DMI.

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<sup>11</sup>In any standard DGE model, each household is assume to solve her optimization problem as the wage and the interest rate are given. Notice that under the situation, rich households will save their asset over their optimal level. See Subsection 3.1.

<sup>12</sup>In standard models with CMI,  $\rho' = 0$ , so  $\rho$  is constant. See also Hirose and Ikeda (2008) for assumptions on  $u$  and  $\rho$  that can be applied to both DMI and IMI.

The Hamiltonian associated with our optimization problem is

$$\mathcal{H} \equiv u(c)X + \lambda(w + ra - c) - \mu\rho(c)X,$$

where  $\lambda$  and  $\mu$  are the co-state variables. The necessary conditions for optimality are

$$\frac{\partial \mathcal{H}}{\partial c} = u'(c)X - \lambda - \mu\rho'(c)X = 0, \quad (3a)$$

$$\frac{\partial \mathcal{H}}{\partial a} = \lambda r = -\dot{\lambda}, \quad (3b)$$

$$\frac{\partial \mathcal{H}}{\partial X} = u(c) - \mu\rho(c) = -\dot{\mu}. \quad (3c)$$

Let  $Z \equiv \lambda/X$  to simplify notation. Then (3a) and (3b) can be rewritten as

$$Z = u'(c) - \mu\rho'(c),$$

$$\dot{Z} = Z[\rho(c) - r].$$

Using the above, our dynamic general equilibrium system can be described as

$$\dot{a} = w(k) + r(k)a - c, \quad (4a)$$

$$\dot{Z} = Z[\rho(c) - r(k)], \quad (4b)$$

$$\dot{\mu} = \mu\rho(c) - u(c), \quad (4c)$$

$$0 = u'(c) - \mu\rho'(c) - Z. \quad (4d)$$

## 2.3 The Steady State

We define the steady state of the model as when all variables for households with asset, i.e.,  $a$  (or  $k$ ),  $Z$ ,  $\mu$  and  $c$ , are constant, and the consumption of households without asset is also constant at  $c^* = w(k)$ . Then the steady state is a solution to the following system of equations<sup>13</sup>

$$0 = w(k) + \frac{r(k)k}{\theta} - c, \quad (5a)$$

$$0 = Z[\rho(c) - r(k)], \quad (5b)$$

$$0 = \mu\rho(c) - u(c), \quad (5c)$$

$$Z = u'(c) - \mu\rho'(c). \quad (5d)$$

These conditions say that at the steady state, consumption must be equal to income (condition (5a)), the interest rate must be equal to the discount factor of households with asset (5b), utility is constant ( $\dot{\mu} = 0$ , (5c)), and (5d) equates the current value of the shadow price to the marginal-utility increase. In what follows, we use “ $\sim$ ” to denote the steady state value of each variable.

More specifically, conditions (5a) and (5b) give the steady state solution pair  $(\tilde{k}, \tilde{c})$ , which can be rewritten as  $c = i(k, \theta)$  and  $k = \kappa(c)$ , where

$$\kappa(c) \equiv r^{-1}(\rho(c)).$$

Due to DMI, the level of capital stock equating the interest rate to the discount factor is

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<sup>13</sup>Our model does not have a “satiated” steady state ( $Z$  being equal to zero) as discussed in Hirose and Ikeda (2008), since our assumption on  $u$  and  $\rho$  ensures that the steady state value of  $Z$  must be positive.

increasing in  $c$ :

$$\frac{d\kappa(c)}{dc} = \frac{\rho'(c)}{r'(k)} > 0. \quad (6)$$

Compare to under CMI, where  $\rho'(c) = 0$ . It follows that condition (6) gives an important result

**Lemma 3** *Under DMI, an increase in the consumption of rich households raises the capital stock in the economy.*

Lemma 3 basically says that by the definition of DMI, an increase in the rich's consumption makes them more patient, which increases their investment, leading to a trickle down effect, through increasing the wage rate and eventually benefiting the poor households.

Once the steady state  $\tilde{c}$  is determined, we see from (5c) and (5d) that

$$\tilde{\mu} = \frac{u(\tilde{c})}{\rho(\tilde{c})} \quad \text{and} \quad \tilde{Z} = \frac{u'(\tilde{c})\rho(\tilde{c}) - u(\tilde{c})\rho'(\tilde{c})}{\rho(\tilde{c})},$$

where  $\tilde{\mu}$  (increasing in  $\tilde{c}$ ) can be interpreted as the steady state level of welfare in the sense that the discounted sum of utility,  $\int_0^\infty u(c)X dt$ , is equal to  $\tilde{\mu}$  when  $c(t) = \tilde{c}$  for  $\forall t \geq 0$ .

Finally we examine the stability and uniqueness of the steady state. Let  $k_1$  and  $k_2$  be the values of the capital stock that equate the interest rate to  $\rho(0)$  and zero respectively:

$$k_1 \equiv r^{-1}(\rho(0)) \quad \text{and} \quad k_2 \equiv r^{-1}(0), \quad (7)$$

where  $k_1 < k_2$  holds.<sup>14</sup> Then, for any  $\theta \in (0, 1]$ ,

$$i(k_1, \theta) = w(k_1) + \frac{\rho(0)k_1}{\theta} > 0,$$

$$i(k_2, \theta) = w(k_2) < \infty.$$

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<sup>14</sup>The Inada conditions:  $\lim_{k \rightarrow 0} f'(k) = \infty$  and  $\lim_{k \rightarrow \infty} f'(k) = 0$  ensure the existence of  $k_1$  and  $k_2$ .

It is apparent from  $d\kappa/dc > 0$ ,  $k_1 = \kappa(0)$ , and  $k_2 = \lim_{c \rightarrow \infty} \kappa(c)$  that there necessarily exists an intersection of the two graphs,  $c = i(k, \theta)$  and  $k = \kappa(c)$ , as in Figure 1.

In the rest of the paper, we assume

**Assumption 1a:** The intersection between  $c = i(k, \theta)$  and  $k = \kappa(c)$  is unique for any  $\theta \in (0, 1]$ ;

Or more strictly,

**Assumption 1b:** Both  $c = i(k, \theta)$  and  $k = \kappa(c)$  are strictly concave in  $k$  and  $c$ , respectively, for any  $\theta \in (0, 1]$ .

**Remark 1** *The strict concavity of  $i$  is satisfied when technology takes the Cobb-Douglas form, while that of  $\kappa$  implies that we basically exclude the extreme case where rich households become much more patient when their income slightly rises.<sup>15</sup> An example of the pair of functions  $f$  and  $\rho$  that satisfies Assumption 1b is provided in Section 4 (see also Assumptions 2 and 3 below).*

Under Assumption 1a, at the intersection, the slope of  $c = i(k, \theta)$  must be smaller than that of  $c = \rho^{-1}(r(k))$ , i.e.,  $\partial i/\partial k < (\partial \kappa/\partial c)^{-1}$ , and hence we have:<sup>16</sup>

$$\frac{(\theta - 1)w'(k) + \rho(c)}{\theta} \rho'(c) > f''(k). \quad (8)$$

Combining this with the stability analysis in Appendix 7.1, we obtain:<sup>17</sup>

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<sup>15</sup>If Assumption 1b is violated, there may be an odd number of steady states. (8) holds in the first steady state, but fails in the second one, ..., and it also holds in the last one. One can verify from the proof of Lemma 3 (Appendix 7.1) that the steady states with (8) are saddle points, while the others are either unstable or indeterminate. Assuming the uniqueness of the steady state, we will focus on a saddle point.

<sup>16</sup>Condition (8) holds at any intersection and hence the steady state is unique, if the following is met:

$$\frac{1}{\rho(0)} \min_{k \in [k_1, k_2]} [-f''(k)] \geq -\frac{\rho'(0)}{\theta} \max_{k \in [k_1, k_2]} \left[ 1 + (1 - \theta) \frac{k f''(k)}{\rho(0)} \right].$$

This inequality degenerates to the bounded slope assumption in Chang (2009), when  $\theta = 1$ . Then the steady state of his model (of endogenous time preference with DMI) is unique and is a saddle point.

<sup>17</sup>Notice that with CMI, the intersection must uniquely exist, and (8) holds at the unique intersection. Also, one can easily verify that Lemma 3 remains valid under CMI.



**Lemma 4** *Under Assumption 1a, the steady state of the dynamic system is a saddle point and unique.*

The above completes the basic setup of the model. In the unique steady state under DMI, rich households save more when their income rises. More interestingly, poor households' welfare increases too, stemming from an increase in the wage rate by Lemma 3 (see also Figure 1).

In addition, we obtain a clear trickle down effect as follows

**Proposition 1** *Under DMI, a decrease in the share of rich households,  $\theta$ , raises poor households' income  $w$  and consumption  $\tilde{c}^*$ , which is absent under CMI.*

This Proposition implies that the smaller is the share of rich households, the higher the income of the poor becomes. This is an effect that does not exist under CMI, where  $k = \kappa(c)$  in Figure 1 becomes a vertical line passing through  $\tilde{k}$ , because  $\tilde{k}$  is only a solution to  $r(k) = \rho$ . Even though a decrease in  $\theta$  shifts up curve  $c$ , raising  $\tilde{c}$ , it does not change  $\tilde{c}^*$ . In contrast, under DMI,  $\rho(c) = r(k)$  becomes the positively sloped curve  $k = \kappa(c)$ , and a higher  $\tilde{c}$  lowers  $r(k)$ , which requires a higher  $\tilde{k}$ , raising  $\tilde{c}^* = w(k)$ .

### 3 Inequality under DMI

We now consider inequality under DMI, by changing the fraction of rich households,  $\theta$ , and examine its impact on the steady state variables. As discussed above, lowering  $\theta$  is favorable for the poor in the sense that it will raise the steady state capital stock and in turn the wage income of poor households. Again, this effect does not exist under CMI.

### 3.1 Unequal but preferred

With DMI, the decrease in  $\theta$  raises the capital stock and the wage in the steady state, implying that some level of inequality may be preferable. Indeed, we can show that an economy with an *uneven* distribution of income among households (i.e., capitalism) is preferable for *all* households to an economy with each household having an identical level of asset (i.e., communism). In contrast, with CMI, the welfare of a typical household under communism is higher than that of poor households though lower than that of rich households under capitalism. Thus, with CMI, communism is good for the poor but bad for the rich, while with DMI, communism is not good for either group.

To see this clearly, suppose all households own the same level of asset, i.e., communism with  $\theta = 1$ . Then their income would be given by

$$\begin{aligned} I(k) &\equiv w(k) + r(k)k \\ &= f(k) - \delta k, \end{aligned}$$

which is increasing in  $k$  with  $k < k_2$ .<sup>18</sup> In Figure 1,  $I(k)$  is represented by the blue curve. It intersects with the curve  $\rho(c) = r(k)$  at point  $E_1$  and crosses the vertical line passing through  $\tilde{k}$  at point  $E_3$ .

Under CMI, from previous sections we have the vertical line passing through  $\tilde{k}$ , and at the steady state, the rich's welfare is at point  $E_1$  but the poor's welfare is at point  $E_2$ . Since  $E_3$  lies above  $E_2$  but below  $E_1$  on the same vertical line, it is clear that a typical household's welfare under communism is higher than that of the poor but lower than that of the rich under capitalism.

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<sup>18</sup>Notice that  $k_2$  corresponds to the Golden Rule level of capital (per household).

Next, under DMI, let  $\tilde{k}(1)$  and  $\tilde{c}(1)$  be the steady state levels of capital and consumption in the economy with  $\theta = 1$ . Then,  $\tilde{k}(1) < k_2$  holds, and hence the income (and consumption) in the steady state must be smaller than  $I(k_2) = w(k_2)$  :

$$\tilde{c}(1) = I(\tilde{k}(1)) < w(k_2).$$

In Figure 1, point  $E_1$  represents the case of  $\theta = 1$  under DMI, which clearly shows that the per capita consumption is lower than  $\tilde{c}^*$  at point  $E_2$ , the poor's steady-state consumption when  $\theta < 1$ . That is, under DMI, communism provides the lowest welfare among all possibilities for everybody.

If there are two types of households and only few households own asset (i.e.,  $\theta$  is sufficiently small), capital accumulates to almost the same level as  $k_2$ , because the very-wealthy households have an extremely low discount rate due to DMI. As such, they will accumulate capital until the interest rate approaches zero. It can be verified as follows. For  $k < k_2$ ,

$$\frac{\partial i(k, \theta)}{\partial \theta} = -\frac{r(k)k}{\theta^2} < 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} i(k, \theta) = \infty.$$

Therefore,

$$\lim_{\theta \rightarrow 0} \tilde{k} = k_2 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \tilde{c} = \infty. \tag{9}$$

Also, notice that

$$\lim_{\theta \rightarrow 0} \tilde{c}^* = w(k_2),$$

which implies

$$\lim_{\theta \rightarrow 0} \tilde{c}^* > \tilde{c}(1).$$

Thus we have<sup>19</sup>

**Proposition 2** *The steady-state welfare in the economy without inequality (i.e., everyone becoming a capitalist with an identical level of asset), is lower than the poor household's welfare in the steady state under a sufficiently small  $\theta$ .*

The Proposition implies that under DMI, some level of inequality is preferable for the whole economy, essentially because rich households are more patient and save more, which lowers the interest rate and in turn raises the wage income of poor households through production linkages. On the contrary, if the capital stock is spread over more capital owners (i.e., lowering inequality), then each capitalist saves and invests less, resulting in lower welfare in the long run. The generated consequences from this Proposition are similar to those of the “Trickle Down Theory” (Aghion and Bolton, 1997), albeit via a starkly different mechanism.

The detailed mechanism behind this surprising result stems from two assumptions, one of which is  $\lim_{c \rightarrow \infty} \rho(c) = 0$ ,<sup>20</sup> and the other is that rich households cannot coordinate with each other on the level of asset holdings, even when their population share is sufficiently small. In contrast, the proposition may not hold, if rich households could strategically behave such that they optimize their discounted sum of utility subject to the following budget constraint,<sup>21</sup>

$$\dot{a} = w(a\theta) + r(a\theta)a - c.$$

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<sup>19</sup>Here, we define the steady state level of welfare for households without asset as  $u(\tilde{c}^*)/\rho(\tilde{c}^*)$ , analogous to that for households with asset.

<sup>20</sup>If  $\lim_{c \rightarrow \infty} \rho(c) > 0$ ,  $\lim_{\theta \rightarrow 0} \tilde{k} < k_2$ , and hence the Proposition above may not hold.

<sup>21</sup>This specification may be justified when rich households notice that  $k = \alpha\theta$  with their coordination on asset holdings, such as when only one household owns all assets in the whole economy. See also Sorger (2008), who considers strategic saving decisions in the Ramsey model.

Then, one of the steady state conditions on  $k$  and  $c$  would change as

$$\rho(c) = r(k) + (\theta - 1)w'(k).$$

In this case, the steady state capital stock would not converge to  $k_2$  when  $\theta$  goes to zero with  $\lim_{\theta \rightarrow 0} \tilde{c} = \infty$  and  $\lim_{c \rightarrow \infty} \rho(c) = 0$ .

The above Proposition also contrasts with Piketty (2014), where inequality increases if the interest rate is higher than the growth rate, since the rich gets richer through investment. However, in our model with DMI, the process does not stop there, because investment by the rich lowers the interest rate and raises the wage income, benefiting the poor eventually with a trickle down.

### 3.2 Income Gaps and the Gini Coefficient

We have shown that lowering  $\theta$  makes poor households better off in the steady state. A related issue is whether the income gap widens or not, which we investigate now.

The incomes of households with and without assets in the steady state are respectively

$$\tilde{I} = w(\tilde{k}) + \frac{\rho(\tilde{c})\tilde{k}}{\theta} \quad \text{and} \quad \tilde{I}^* = w(\tilde{k})$$

We can define the income gap in terms of both *level* and *ratio* differences, respectively as:

$$g \equiv \tilde{I} - \tilde{I}^* = \frac{\rho(\tilde{c})\tilde{k}}{\theta}$$

and  $\hat{g} \equiv \frac{\tilde{I}}{\tilde{I}^*} = 1 + \frac{\rho(\tilde{c})\tilde{k}}{\theta w(\tilde{k})}$ .

Notice that when the share  $\theta$  decreases,  $\tilde{I}$  and  $\tilde{I}^*$  will rise along the graphs,  $k = \kappa(c)$  and

$c = w(k)$ , respectively. Therefore, if  $\kappa$  and  $w$  are strictly concave in  $c$  and  $k$  respectively as in Figure 1, then both  $g$  and  $\hat{g}$  are decreasing in  $\theta$ : Income gaps will rise, as the share of rich households decreases. Also, notice that  $\kappa'' < 0$  corresponds to the case where the effect of DMI is not strong.

Then the Gini coefficient can be calculated as,

$$\begin{aligned} G &= 1 - \frac{\tilde{I}^*(1-\theta)^2/2 + \tilde{I}^*\theta(1-\theta) + \tilde{I}\theta^2/2}{[\tilde{I}^*(1-\theta) + \tilde{I}\theta]/2} \\ &= \frac{\theta\hat{g}}{\theta\hat{g} + 1 - \theta} - \theta, \end{aligned}$$

which gives:

**Lemma 5** *The Gini coefficient  $G$  is increasing in the ratio income-gap  $\hat{g}$ .*

We are now in a position to state

**Proposition 3** *Under Assumption 1b, as the share of rich households increases, both the level and the ratio income-gaps narrow. However, there is a non-monotonic relationship between the share  $\theta$  and the Gini coefficient  $G$ :  $G = 0$  holds when either  $\theta$  approaches 0 or equals 1;  $\frac{dG}{d\theta} > 0$  when  $\theta$  is sufficiently small, but  $\frac{dG}{d\theta} < 0$  for  $\theta \geq 1/2$ .*

**Proof.** It is apparent from Figure 1 that both  $g$  and  $\hat{g}$  are decreasing in  $\theta$  when  $\kappa$  is strictly concave in  $c$ . And  $G = 0$  holds when  $\theta$  goes to 0, because

$$\lim_{\theta \rightarrow 0} \theta \tilde{I} = \lim_{\theta \rightarrow 0} \left[ \theta w(\tilde{k}) + \rho(\tilde{c})\tilde{k} \right] = 0.$$

Differentiation gives,

$$\begin{aligned} G_\theta &= \frac{\hat{g} + (1 - \theta)\theta\hat{g}_\theta}{(\theta\hat{g} + 1 - \theta)^2} - 1 \\ &= \frac{(1 - 2\theta)(\hat{g} - 1) + (1 - \theta)\theta\hat{g}_\theta - [\theta(\hat{g} - 1)]^2}{(\theta\hat{g} + 1 - \theta)^2}, \end{aligned}$$

where  $\hat{g}_\theta < 0$ . Since  $\hat{g} > 1$ ,  $G_\theta$  must be negative for  $\theta \geq 1/2$ ; but it can be positive for a sufficiently small  $\theta$ ,<sup>22</sup> because  $G > 0$  for any  $\theta \in (0, 1)$  and  $\lim_{\theta \rightarrow 0} G = 0$ . ■

Thus, under DMI, the Gini coefficient  $G$  does not monotonically increase. Instead, the Proposition implies that the Gini coefficient exhibits a sharp *inverted-U shape*, first increasing then decreasing, following a fall in the percentage owning assets,  $\theta$ . It again stems from DMI and its trickle down effect, as discussed extensively before. The result stands in sharp contrast to the case of CMI, where  $G$  increases linearly as  $\theta$  goes to zero.<sup>23</sup>

This relationship is similar to the Kuznets curve (Kuznets, 1955), and it may be good news for developing countries such as China, India and Latin America, who are trying to catch up with the developed countries.

The above Proposition has important implications. In order to reduce inequality, Piketty (2014) proposes that a progressive annual global wealth tax of up to 2%, combined with a progressive income tax reaching as high as 80%. While in the present model, as the fraction of population owning assets rises to above a certain level, inequality gradually falls, even without any government intervention.

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<sup>22</sup>Indeed, one can verify that under Assumptions 2 and 3,  $\lim_{\theta \rightarrow 0} \theta\hat{g}_\theta/(\hat{g} - 1) = -(1 + \beta)^{-1} > -1$ .

<sup>23</sup>In the case of CMI,  $G = \rho\tilde{k}(1 - \theta)/[w(\tilde{k}) + \rho\tilde{k}]$ , where  $\tilde{k}$  does not depend on  $\theta$ .

## 4 Income Transfer and Technology Advancement

In this section, we examine how income transfer by a tax on capital income and technology advancement affect the steady state. Precisely due to the impact of DMI, any change that makes households with asset richer also has a positive effect on the steady state income of households without asset, as we shall clearly demonstrate below.

Political economy models, in their simplest version, start with the premise that inequality leads to redistribution and then it is argued that redistribution hurts growth. For versions of this argument see Alesina and Rodrik (1994), Persson and Tabellini (1991), Benhabib and Rustichini (1998). Indeed, redistribution mostly narrows income gaps also in our model.<sup>24</sup>

However, an income transfer from the rich to the poor may lower not only the steady state capital stock, *but also the steady state welfare of poor households*, though they are directly benefited by the transfer. This will happen if the share  $\theta$  is small, when the positive direct effect of transfer on poor households' income is dominated by the indirect effect through the fall in the steady state capital stock. Thus,  $\theta$  is a key parameter that determines the opposite effects of the transfer.

Specifically, we introduce a 'capital tax'  $\tau$  on the asset income of rich households, intended to reduce the income gap. The tax revenue is used as a lump-sum transfer,  $T$ , to poor households. The post-tax income for rich households is then

$$I = w + (1 - \tau)ra,$$

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<sup>24</sup>See also Proposition 3 in Iwasa and Zhao (2013).



while that for poor households becomes

$$I^* = w + T,$$

for which the government budget constraint  $\tau r a \theta = T(1 - \theta)$  holds. Also, it is natural to assume the income of rich households to be higher than that of poor households,  $I \geq I^*$ , for which  $\tau \leq 1 - \theta$  is required.

Now, the steady state is a solution to the following system of equations

$$0 = w(k) + \frac{1 - \tau}{\theta} r(k)k - c, \quad (10a)$$

$$0 = Z[\rho(c) - (1 - \tau)r(k)], \quad (10b)$$

$$0 = \mu\rho(c) - u(c), \quad (10c)$$

$$Z = u'(c) - \mu\rho'(c). \quad (10d)$$

Introducing  $\tau$  does not change the model, except that the interest rate is given by  $(1 - \tau)r$ .

And the steady state solution pair  $(\tilde{k}, \tilde{c})$  is given by the intersection between  $c = i(k, \theta, \tau)$  and  $k = \kappa(c, \tau)$ , where

$$i(k, \theta, \tau) \equiv w(k) + \frac{1 - \tau}{\theta} r(k)k,$$

$$\kappa(c, \tau) \equiv r^{-1} \left( \frac{\rho(c)}{1 - \tau} \right).$$

In the rest of the paper, we assume<sup>25</sup>

**Assumption 1A:** The intersection between  $c = i(k, \theta, \tau)$  and  $k = \kappa(c, \tau)$  is unique for any

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<sup>25</sup> Assumptions 2 and 3 in this section are sufficient for Assumption 1B also in this case.

pair  $(\theta, \tau)$  with  $\theta \in (0, 1]$  and  $\tau \leq 1 - \theta$ ;

Or more strictly,

**Assumption 1B:** Both  $c = i(k, \theta, \tau)$  and  $k = \kappa(c, \tau)$  are strictly concave in  $k$  and  $c$ , respectively, for any pair  $(\theta, \tau)$  with  $\theta \in (0, 1]$  and  $\tau \leq 1 - \theta$ .

At the unique intersection, we have

$$\frac{(\theta - 1 + \tau)w'(k) + \rho(c)}{\theta}\rho'(c) > (1 - \tau)f''(k). \quad (11)$$

## 4.1 Steady State Welfare

We next consider the effects of changes in  $A$  and  $\tau$  on the steady state level of welfare for both types of households.<sup>26</sup> Totally differentiating  $c = i(k, \theta, \tau)$  and  $k = \kappa(c, \tau)$  with respect to  $A$  and  $\tau$  to give

$$\begin{aligned} & \begin{pmatrix} -\frac{1-\tau-\theta}{\theta}\tilde{k}f'' - \frac{\rho}{\theta} & 1 \\ (1-\tau)f'' & -\rho' \end{pmatrix} \begin{pmatrix} dk \\ dc \end{pmatrix} \\ &= \begin{pmatrix} \frac{f}{A} + \frac{1-\tau-\theta}{A\theta}\tilde{k}f' & -\frac{\rho\tilde{k}}{(1-\tau)\theta} \\ -\frac{1-\tau}{A}f' & \frac{\rho}{1-\tau} \end{pmatrix} \begin{pmatrix} dA \\ d\tau \end{pmatrix}, \end{aligned}$$

where each element of the matrices is evaluated at the steady state. Then, we obtain:

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<sup>26</sup>Here,  $A > 0$  is the total productivity with  $f(k) = Af(k)$  for  $\forall k$ .

**Lemma 6** *Under Assumption 1,*

$$\frac{\partial \tilde{k}}{\partial A} = \frac{-\theta \rho' f + [(1 - \tau)\theta - (1 - \tau - \theta)\rho' \tilde{k}] f'}{AD\theta} > 0, \quad (12a)$$

$$\frac{\partial \tilde{c}}{\partial A} = \frac{(1 - \tau)(\rho f' - \theta f f'')}{AD\theta} > 0, \quad (12b)$$

$$\frac{\partial \tilde{k}}{\partial \tau} = -\frac{\rho(\theta - \rho' \tilde{k})}{D(1 - \tau)\theta} < 0, \quad (12c)$$

$$\frac{\partial \tilde{c}}{\partial \tau} = -\frac{\rho(\rho - \theta \tilde{k} f'')}{D(1 - \tau)\theta} < 0, \quad (12d)$$

where

$$D \equiv \left( \frac{1 - \tau - \theta}{\theta} \tilde{k} f'' + \frac{\rho}{\theta} \right) \rho' - (1 - \tau) f'' > 0 \quad \text{and} \quad (1 - \tau)\theta - (1 - \tau - \theta)\rho' \tilde{k} > 0$$

hold from (11).

From the Lemma, one sees that the steady state capital stock increases when technology  $A$  improves, which must in turn raise the steady state level of welfare for all households, since

$$c^* = i^*(k, \theta, \tau) \equiv w(k) + \frac{\tau r(k)k}{1 - \theta}$$

and  $\partial i^* / \partial k > 0$ .

On the other hand, the effect of  $\tau$  on poor households' welfare is ambiguous, because it has a negative effect on the steady state capital stock, and hence it may reduce poor households' income including transfers.<sup>27</sup>

As a particularly interesting finding, it is possible for the 'capital tax' on the rich to lower the steady state welfare of poor households, especially in an economy where almost all households

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<sup>27</sup>Hamada (1967) considers the effect of transfers between capitalists and workers on the latter's income on the equilibrium growth path with constant saving ratios, and he finds the optimal tax rate is zero,  $\tau = 0$ .

are poor and unable to own asset (e.g., a low  $\theta$ ). In such a case, the government can increase the steady state welfare for all households by setting  $\tau$  to be negative, in effect subsidizing the rich and taxing the poor!

To be more specific, partially differentiating  $\tilde{c}^*$  with respect to  $\tau$  yields

$$\frac{\partial \tilde{c}^*}{\partial \tau} = \frac{\rho(\tilde{c})\tilde{k}}{(1-\theta)(1-\tau)^2} + w'(\tilde{k})\frac{\partial \tilde{k}}{\partial \tau} + \frac{\tau\rho(\tilde{c})}{(1-\theta)(1-\tau)} \cdot \frac{\partial \tilde{k}}{\partial \tau} + \frac{\tau\tilde{k}\rho'(\tilde{c})}{(1-\theta)(1-\tau)} \cdot \frac{\partial \tilde{c}}{\partial \tau},$$

where the first term on the RHS denotes a positive direct effect, and the rest is the sum of several indirect effects: the second and third terms are negative, which come from the fact that an increase in  $\tau$  reduces the capital stock, and hence both the wage rate and the amount of transfer fall, while the last term is positive due to a rise in the rental rate under DMI.

From Lemma 6, we see

$$\begin{aligned} \frac{\partial \tilde{c}^*}{\partial \tau} &= \hat{D} \left\{ \tilde{k} \left[ (1-\tau-\theta)\rho'\tilde{k}f'' + \rho\rho' - (1-\tau)f''\theta \right] + (1-\theta)(1-\tau)\tilde{k}f''(\theta - \rho'\tilde{k}) \right. \\ &\quad \left. - \tau\rho(\theta - \rho'\tilde{k}) - \tau\tilde{k}\rho'(\rho - \theta\tilde{k}f'') \right\} \\ &= \hat{D} \left[ \theta(1-\tau)\rho'\tilde{k}^2f'' - \theta\rho'\tilde{k}^2f'' + \rho\rho'\tilde{k} - \theta^2(1-\tau)\tilde{k}f'' \right. \\ &\quad \left. - \tau\theta\rho + \tau\theta\rho'\tilde{k}^2f'' \right] \\ &= \hat{D} \left[ \rho\rho'\tilde{k} - \theta^2(1-\tau)\tilde{k}f'' - \tau\rho\theta \right], \end{aligned} \tag{13}$$

where  $\hat{D} \equiv \rho/D\theta(1-\theta)(1-\tau)^2$ .

We summarize the above results as

**Proposition 4** *Let Assumption 1A hold. Then, a positive capital-income tax on rich households raises the welfare of the poor under CMI, but may not do so under DMI: for a sufficiently small  $\theta$ , subsidizing the rich and taxing the poor may raise all households' welfare in the steady*

state under DMI.<sup>28</sup>

**Proof.** See Appendix 7.2. ■

Note that this result can be much more clearly demonstrated, if we specify as follows:

**Assumption 2:** The production technology takes the Cobb-Douglas form:  $f(k) = Ak^\alpha$ ,  $\alpha \in (0, 1)$ ;

and

**Assumption 3:**  $\rho(c) = B(c + 1)^{-\beta}$ ,  $\beta \in (0, 1 - \alpha)$ ,<sup>29</sup>

where Assumption 3 implies that as capital accumulates, the interest rate falls faster than the speed at which rich households become patient (as their consumption levels rise), i.e.,

$$\left| \frac{d\rho}{\rho} \bigg/ \frac{d(c+1)}{c+1} \right| < \left| \frac{dR}{R} \bigg/ \frac{dk}{k} \right|.$$

Under Assumptions 2 and 3, which imply that the effect of DMI is weak, we find that for  $\tau \in (-\beta, 1 - \theta)$ , reducing the capital tax or raising the capital subsidy increases the steady state welfare of poor households in an economy with some  $\theta$ , and the welfare is maximized at some  $\tau \in (-\beta, 0)$ ,<sup>30</sup> rather than by an income transfer to them!

The results under DMI contrast sharply with those under CMI. Poor households' steady state income is more likely to decrease under DMI than under CMI, when the government raises the tax on rich households (the first term in (13)). That is, a positive capital tax on rich households (and transfer to the poor) is preferable for poor households under CMI by increasing their present and future incomes; But under DMI, it is possible that  $\partial \tilde{c}^* / \partial \tau < 0$  for

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<sup>28</sup>For a sufficiently high  $\theta$ , a positive tax raises the welfare of the poor under DMI as discussed below.

<sup>29</sup>One can verify that Assumption 3 is consistent with (2).

<sup>30</sup>See the proof of Proposition 4 in Appendix 7.2.

$\tau \geq 0$ . Although the effect of DMI is not strong, subsidizing the rich and taxing the poor will raise poor households' welfare in the steady state if there exists a large scale of inequality in the economy.

The above results remind us of the Chinese experience in the past 35 years. Until the early 1980s, most Chinese were very poor and owned almost zero assets. The government opened Special Economic Zones to allow business owners (a tiny fraction of the population then) to do business tax free, and with land, import or export and other subsidies. Thus came Deng Xiaoping's famous talk to "allow a small fraction of the population get rich first". Recent studies such as Chang, Gu and Tam (2015) and Gu, Li and Tam (2015) find that inequality is a big mover for the savings glut in China, which is responsible for its investment-dependent growth for the past several decades. It is especially worth mentioning that even North Korea opened Special Economic Zones along the borders with both China and South Korea.

On the other hand, in a more mature economy with a higher  $\theta$  (a higher fraction of population owning assets), then under Assumptions 2 and 3,<sup>31</sup>

$$\frac{\partial \tilde{c}^*}{\partial \tau} > 0 \text{ for any feasible } \tau \leq 0;$$

that is, some positive capital tax will maximize poor households' steady state welfare. This arises because (13) yields

$$\frac{\partial \tilde{c}^*}{\partial \tau} = \hat{D}\theta \left\{ \left[ \left( -\frac{\tau}{\theta} \tilde{k} f'' + \frac{\rho}{\theta} \right) \rho' - (1 - \tau) f'' \right] \tilde{k} - \tau \left( \rho - \frac{\rho' \tilde{k}^2 f''}{\theta} \right) + (1 - \theta)(1 - \tau) \tilde{k} f'' \right\},$$

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<sup>31</sup>Here, "feasible" means  $i^*(\tilde{k}, \theta, \tau) = w(\tilde{k}) + \tau r(\tilde{k})\tilde{k}/(1 - \theta) \geq 0$ , i.e.,  $\tau \geq -(1 - \theta)w(\tilde{k})/r(\tilde{k})\tilde{k}$ , the right-hand side of which goes to zero when  $\theta$  goes to 1.

where, if  $\theta$  is sufficiently close to 1, (i). the term in [...] is positive from  $D > 0$ ; (ii)  $\rho > \rho' \tilde{k}^2 f''$  holds under Assumptions 2 and 3, and hence the second term is also non-negative given  $\tau \leq 0$ ; (iii) the first term in braces {...} dominates the last term when  $\theta$  goes to 1 (see footnote 30). Thus, we see that when  $\theta$  is sufficiently high, the tax on capital income will increase the welfare of the poor.

Intuitively, this occurs because when  $\theta$  is sufficiently high, there are enough rich people in the economy, and thus it is relatively easier to raise the welfare of the poor by taxing the rich, as might be the case for China nowadays, when the fraction of the rich has reached a certain high level, especially in some big cities and along the coast.

Using a numerical example, we demonstrate that there is a critical level of  $\theta$  under which the effect of  $\tau$  on poor households is negative, and vice versa.

## 4.2 Income Gaps

We have considered the effect of capital tax on welfare and shown that when positive productivity shock occurs, all households become better off at the steady state. A related issue is whether their income gap widens or not, which we investigate now.<sup>32</sup>

The incomes of households with and without assets at the steady state are respectively

$$\tilde{I} = w(\tilde{k}) + \frac{\rho(\tilde{c})\tilde{k}}{\theta}$$

and  $\tilde{I}^* = w(\tilde{k}) + \frac{\tau\rho(\tilde{c})\tilde{k}}{(1-\theta)(1-\tau)},$

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<sup>32</sup>We omit the effect of  $\tau$  on the income gaps, which can be divided into the sum of a direct effect and an indirect effect. The indirect effect can be negative under strong DMI, when a tax hike on rich households reduces the capital stock by a large scale, and hence the wage rate falls sharply. Then it is possible for the income gap to widen, when the indirect effect dominates the direct one. However, the tax mostly narrows the gaps as in the case of CMI.

from which we have the income gaps as

$$g = \frac{1 - \theta - \tau}{\theta(1 - \theta)(1 - \tau)} \rho(\tilde{c})\tilde{k}$$

$$\text{or } \hat{g} = \frac{\theta + \frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}}{\theta + \frac{\theta\tau}{(1-\theta)(1-\tau)} \cdot \frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})}},$$

where  $g$  and  $\hat{g}$  (and hence  $G$ ) are increasing in  $\rho(\tilde{c})\tilde{k}$  and  $\rho(\tilde{c})\tilde{k}/w(\tilde{k})$  respectively.

Calculations give that under Assumptions 1A and 2,

$$\frac{\partial}{\partial A} [\rho(\tilde{c})\tilde{k}] = \frac{\rho[\rho + (1 - \tau)\delta]}{AD} \left[ 1 + \frac{(1 - \alpha)\delta}{\alpha\rho} \rho' \tilde{k} \right], \quad (14a)$$

$$\frac{\partial}{\partial A} \left[ \frac{\rho(\tilde{c})\tilde{k}}{w(\tilde{k})} \right] = \frac{(1 - \tau)\delta\{(1 - \tau)\alpha\rho + \theta(1 - \alpha)[\rho + (1 - \tau)\delta]\}}{AD\theta(1 - \alpha)[\rho + (1 - \tau)\delta]} \rho' < 0, \quad (14b)$$

Therefore, we have

(i). under CMI,

$$g_A > 0, \quad \text{and } G_A = 0; \quad (15a)$$

(ii). under DMI,

$$G_A < 0, \quad (15b)$$

where  $g_A \equiv \partial g / \partial A$  and  $G_A \equiv \partial G / \partial A$ , respectively.

Some explanation is in order. In (i), under CMI, there is no trickle down effect, and the income gap will widen when technology advancement occurs; while in (ii) under DMI, poor households benefit more from positive shocks, which makes rich households more patient and hence accumulate more capital. It follows that the Gini coefficient is reduced by a rise in productivity. Also, there is a possibility that technology improvement narrows the gap  $g$  under



DMI. However, with Assumption 3, we have  $g_A > 0$ , as in the case of CMI.

**Proposition 5** *Given Assumptions 1A and 2, (i). An increase in productivity  $A$  reduces the Gini coefficient  $G$  under DMI; (ii). It widens the gap  $g$  when DMI is not strong (i.e., when Assumption 3 holds).*

**Proof.** See Appendix 7.3. ■

Acemoglu (2002) argues that the widening of income gap is a result of technology improvement over the past century. Our results on the increase of  $A$  partly reconfirm his prediction: It is only true for the gap in terms of income level ( $g_A > 0$ ), and with regards to the Gini coefficient  $G$ , only true under CMI, but not under DMI as shown in Proposition 5 ( $G_A < 0$ ).

Also, Proposition 5 to some extent justifies the 'capital tax' on rich households, which is similar to what Piketty (2014) proposes. However, it has a negative effect on the steady state capital stock, and hence it may reduce poor households' income including transfers. Then it is necessary to look into the trade-off between the level of welfare and the inequality among households. From Proposition 4, the poor's welfare can be increased when  $\theta$  is sufficiently high, but may be decreased when  $\theta$  is sufficiently low.

## 5 Immigration

We next extend the model to examine the effects of immigration. International labor migration is an integral part of our globalized economy. Workers migrate from poor to rich countries, to seek better opportunities and earn higher income. According to the United Nations,<sup>33</sup> the world stock of migrants reached 243 million in 2015. In this section, we look into how international migration affects the economy in terms of inequality and welfare.

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<sup>33</sup>U.N. Trends in International Migrant Stock: the 2015 Revision.

Consider a developed home country that has both rich and poor households. Poor households (without asset) from a foreign country immigrate into this country, which causes a decrease in the share of rich households  $\theta$ . Applying Proposition 3, we have

**Proposition 6** *Let  $\tau \in [0, 1 - \theta)$  and Assumption 1B hold. Immigration (lowering  $\theta$ ) widens both the level and the ratio income-gap under DMI as well as under CMI.*

This Proposition implies that immigration may cause polarization in the rich countries that host immigrants, which is arguably one of the most important reasons for Brexit in the EU and Donald J. Trump taking over the U.S. presidency.

More interestingly, immigration necessarily brings a positive effect on the steady state welfare of all households when  $\tau = 0$ , because  $\partial \tilde{k} / \partial \theta < 0$ . However, when  $\tau$  is positive the effect will be complicated and depends on the values of  $\theta$  and  $\tau$ . This is because if  $\tau$  is positive, a fall in  $\theta$  implies a decrease in the amount of income transfer,  $T = \tau r(k)k / (1 - \theta)$ , for a fixed  $k$ . Thus, we obtain

**Proposition 7** *Given Assumption 1A, under DMI, immigration (i) raises the steady state welfare of all households in the home country, if  $\theta$  or  $\tau$  is sufficiently small; (ii) lowers (raises) the steady state welfare of poor (rich) households in this country, if  $\theta$  is sufficiently close to 1 and  $\tau$  is sufficiently close to  $1 - \theta$ . In contrast, under CMI, it has no (a negative) effect on the steady state welfare of poor households when  $\tau = 0$  ( $\tau > 0$ ), though it raises that of rich households.*

**Proof.** See Appendix 7.4. ■

To gain an intuitive explanation for Proposition 7, we begin with the case that  $\tau = 0$ , and

the economy has reached a steady state such that

$$\tilde{k} = \frac{K}{L}, \quad \tilde{a} = \frac{K}{\theta L}, \quad \tilde{c} = w \left( \frac{K}{L} \right) + \rho \frac{K}{\theta L}, \quad \text{and} \quad \tilde{c}^* = w \left( \frac{K}{L} \right),$$

where  $\rho = r(K/L)$ .

As more immigrants move in, the labor input  $L$  rises to  $L'$  (the ratio  $\theta$  falls to  $\theta'$ ), and hence the wage rate falls and the capital rental rises. The whole economy thus accumulates capital and will eventually reach a new steady state, which can be characterized by

$$\tilde{k} = \frac{K'}{L'}, \quad \tilde{a} = \frac{K'}{\theta' L'}, \quad \tilde{c} = w \left( \frac{K'}{L'} \right) + \rho \frac{K'}{\theta' L'}, \quad \text{and} \quad \tilde{c}^* = w \left( \frac{K'}{L'} \right),$$

where  $\rho = r(K'/L')$ , and the rich households own more asset than in the old steady state.

Under CMI,  $K'/L' = K/L$  must hold, and hence rich households' steady state consumption will rise, while that of poor households does not change. As a consequence, existing poor households are harmed by immigration due to the short-term fall in the wage rate.

Under DMI, however, an increase in the capital labor ratio ( $K'/L' > K/L$ ) is accompanied by an increase in the consumption level of rich households since they become more patient, and therefore poor households will be made better off in the long run through the rich's added investment. In other words, DMI generates a further effect above CMI, which in this case is a positive trickle-down, from the haves to the have-nots.

Notice that Propositions 6 and 7 (i) together imply that immigration that adds poor people to the host country, increases the steady-state welfare of all households in the country due to capital accumulation stimulated by labor migration, but it also widens the income gap between the rich and the poor. These are consistent with observations in North America and Europe

which have experienced increases in immigration and average income, but the income gap also rises drastically, as documented in the Introduction section. We have also examined the above effects of immigration on the poor by a numerical example (results available upon request).

## 6 Concluding Remarks

In an economy with intrinsic inequality to begin with, we have examined how endogenous time preference affects social inequality, with special focus on DMI. Our analysis has shown that (i) poor households tend to benefit more from positive shocks under DMI than under CMI; (ii) positive shocks widen the income difference between the rich and the poor when the effect of DMI is small; (iii) inequality may be a necessary evil, in order for a country to increase its welfare faster, especially under DMI. We have also extended the basic model to examine the effects of immigration into rich countries, and found that it increases the average welfare on the one hand, but enlarges the income gap on the other hand.

Our result that increasing inequality (a fall in  $\theta$ ) makes the poor households better off is derived in the absence of international credit market. If international lending and borrowing are available, this result may be altered, mainly because international borrowing may change the asset holdings ratio  $\theta$ .

Finally on an international scale, patience may impact rich and poor countries differently in terms of income inequality. Developing countries use inferior technology and their capital markets are less efficient and less complete, leading to lower income. An increase in patience may raise the income inequality because rich households tend to gain more asset income (higher interest rates) when capital markets are less complete. Examples are again China and India, where savings and investment are high, but the income inequality is increasing rapidly. In

contrast in developed countries, the income inequality may fall when households become more patient, for which case Luxembourg and Switzerland are perhaps good examples.

All these remain interesting issues to be examined in the future.

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## 7 Appendix

### 7.1 Proof of Lemma 3

We evaluate the elements of a Jacobian in the system (equations (4a)–(4d)) to study the local dynamics around the steady state. Here, we introduce the capital tax  $\tau$  in the system to show that the stability of the steady state does not change. Differentiation gives,



$$\begin{aligned}
J(x) &= \det[J - xI] \\
&= \det \begin{bmatrix} [\theta - (1 - \tau)]w' + \rho - x & 0 & 0 & -1 \\ -\theta Z(1 - \tau)r' & -x & 0 & Z\rho' \\ 0 & 0 & \rho - x & -Z \\ 0 & -1 & -\rho' & -M \end{bmatrix} \\
&= \Gamma(x, \theta, \tau),
\end{aligned}$$

where  $M \equiv -u'' + \mu\rho'' > 0$  and

$$\begin{aligned}
\Gamma(x, \theta, \tau) &\equiv Mx^3 - M\{[\theta - (1 - \tau)]w' + 2\rho\}x^2 + (M\rho\{[\theta - (1 - \tau)]w' + \rho\} \\
&\quad - Z[\rho\rho' - \theta(1 - \tau)r'])x + \rho Z\{[\theta - (1 - \tau)]w'\rho' + \rho\rho' - \theta(1 - \tau)r'\}.
\end{aligned}$$

Notice that (11) implies  $\Gamma(0, \theta, \tau) > 0$ . This characteristic equation can be used to derive the local dynamics of the system in the neighborhood of the steady state.

It is clear from  $M > 0$  and  $J(0) = \Gamma(0, \theta, \tau) > 0$  that  $J(x) = 0$  has at least one negative root,  $x_1$ . If  $[\theta - (1 - \tau)]w' + 2\rho = 0$ , then

$$\begin{aligned}
J(x) &= Mx^3 + J'(0)x + J(0) \\
&= M(x - x_1) \left[ x^2 + x_1x - \frac{J(0)}{Mx_1} \right].
\end{aligned}$$

Thus the other two roots,  $x_2$  and  $x_3$ , satisfy  $x_2 + x_3 = -x_1 > 0$  and  $x_2x_3 = -J(0)/Mx_1 > 0$ , which implies that they have positive real parts.

Suppose that  $[\theta - (1 - \tau)]w' + 2\rho \neq 0$ . Then, applying Routh's (1905) theorem, the number

of the roots of  $J(x) = 0$  with positive real parts equals the number of changes in signs in the following sequence:

$$M, \quad -M\{[\theta - (1 - \tau)]w' + 2\rho\}, \quad \frac{\Gamma([\theta - (1 - \tau)]w' + 2\rho, \theta, \tau)}{[\theta - (1 - \tau)]w' + 2\rho}, \quad \Gamma(0, \theta, \tau).$$

(i).  $[\theta - (1 - \tau)]w' + 2\rho > 0$ . Then, the number of changes in signs is two irrespective of the sign of the third term.

(ii).  $[\theta - (1 - \tau)]w' + 2\rho < 0$ . Then,  $[\theta - (1 - \tau)]w' + \rho < 0$  and

$$\begin{aligned} & \frac{\Gamma([\theta - (1 - \tau)]w' + 2\rho, \theta, \tau)}{[\theta - (1 - \tau)]w' + 2\rho} \\ &= \frac{M\rho\{[\theta - (1 - \tau)]w' + \rho\}\{[\theta - (1 - \tau)]w' + 2\rho\} + Z\theta(1 - \tau)r'\{[\theta - (1 - \tau)]w' + \rho\} - Z\rho^2\rho'}{[\theta - (1 - \tau)]w' + 2\rho} \end{aligned}$$

$< 0$ ,

which implies that the number of changes is two.

## 7.2 Proof of Proposition 4

Since

$$\frac{\partial \tilde{c}^*}{\partial \tau} = \frac{\rho\theta}{D(1 - \theta)(1 - \tau)^2} \left[ \frac{\rho\rho'\tilde{k}}{\theta^2} - (1 - \tau)\tilde{k}f'' - \frac{\tau\rho}{\theta} \right], \quad (16)$$

for any  $\tau \leq 0$ ,

$$\frac{\partial \tilde{c}^*}{\partial \tau} > 0 \quad (17)$$

holds in the case of CMI ( $\rho' = 0$ ).

We now show that if  $\theta$  is sufficiently small, then the first term in square brackets of (16)

dominates the second one, and hence we have

$$\frac{\partial \tilde{c}^*}{\partial \tau} < 0 \text{ for } \tau \in [0, 1 - \theta] \quad (18)$$

under  $\rho' < 0$ .

First, we see from (9) that

$$\lim_{\theta \rightarrow 0} \tilde{k} f''(\tilde{k}) = k_2 f''(k_2) \quad \text{and} \quad \lim_{\theta \rightarrow 0} \rho(\tilde{c}) = 0. \quad (19)$$

Also, we obtain  $\partial \tilde{k} / \partial \theta$  and  $\partial \tilde{c} / \partial \theta$  from totally differentiating  $c = i(k, \theta, \tau)$  and  $k = \kappa(c, \tau)$  with respect to  $\theta$  as follows:

$$\frac{\partial \tilde{k}}{\partial \theta} = \frac{\rho \rho' \tilde{k}}{D\theta^2} < 0, \quad (20a)$$

$$\frac{\partial \tilde{c}}{\partial \theta} = \frac{(1 - \tau) \rho \tilde{k} f''}{D\theta^2} < 0. \quad (20b)$$

Using L'Hôpital's rule, we have

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{\rho(\tilde{c})}{\theta} &= \lim_{\theta \rightarrow 0} \rho' \frac{\partial \tilde{c}}{\partial \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{(1 - \tau) \tilde{k} f''}{\frac{\theta}{\rho} (1 - \tau - \theta) \tilde{k} f'' + \theta - \frac{\theta^2}{\rho \rho'} (1 - \tau) f''}. \end{aligned} \quad (21)$$

However, (9) and the fact that  $\tilde{c} = w(\tilde{k}) + \rho(\tilde{c})\tilde{k}/\theta$  together imply

$$\lim_{\theta \rightarrow 0} \frac{\rho(\tilde{c})}{\theta} = \infty. \quad (22)$$

From (21) and (22), we have

$$\lim_{\theta \rightarrow 0} \frac{\rho(\tilde{c})\rho'(\tilde{c})}{\theta^2} = -\infty. \quad (23)$$

If  $\theta$  is sufficiently small, then (18) holds, which proves Proposition 4.

Finally, under Assumptions 2 and 3, we have

$$\begin{aligned} \lim_{\theta \rightarrow 0} \left[ -\frac{\rho'(\tilde{c})\tilde{k}}{\theta} \right] &= \lim_{\theta \rightarrow 0} \frac{\beta\tilde{k}(1-\tau)r(\tilde{k})}{\theta(\tilde{c}+1)} \\ &= \lim_{\theta \rightarrow 0} \frac{\beta\tilde{k}(1-\tau)r(\tilde{k})}{\theta w(\tilde{k}) + \tilde{k}(1-\tau)r(\tilde{k}) + \theta} \\ &= \beta. \end{aligned} \quad (24)$$

Since (16) yields

$$\frac{\partial \tilde{c}^*}{\partial \tau} = -\frac{\rho\theta}{D(1-\theta)(1-\tau)^2} \left[ (1-\tau)\tilde{k}f'' + \frac{\rho}{\theta} \left( \tau - \frac{\rho'\tilde{k}}{\theta} \right) \right],$$

we see from (22) and (24) that for any  $\tau > -\beta$ , there exists some  $\theta$  such that

$$\frac{\partial \tilde{c}^*}{\partial \tau} < 0.$$

Also, if  $\tau \leq -\beta$ , then  $\partial \tilde{c}^*/\partial \tau > 0$  holds for any  $\theta$ : the value of  $\tau$  that maximizes the steady

state levels of welfare for poor households must be greater than  $-\beta$ . This is because

$$\begin{aligned}
\frac{\partial}{\partial \theta} \left[ \frac{\rho'(\tilde{c})\tilde{k}}{\theta} \right] &= \frac{(\rho'' \frac{\partial \tilde{c}}{\partial \theta} \tilde{k} + \rho' \frac{\partial \tilde{k}}{\partial \theta})\theta - \rho' \tilde{k}}{\theta^2} \\
&= \frac{(\rho')^2 \tilde{k} f''}{D\theta^3} \left[ \frac{\rho \rho''}{(\rho')^2} (1-\tau)\tilde{k} - (1-\tau-\theta)\tilde{k} + \frac{\theta(1-\tau)}{\rho'} \right] \\
&= \frac{(\rho')^2 \tilde{k} f''}{D\theta^3 \beta \rho} \left( [(\beta+1)(1-\tau) - \beta(1-\tau-\theta)](1-\tau)(\alpha A \tilde{k}^\alpha - \delta \tilde{k}) \right. \\
&\quad \left. - (1-\tau)\{A \tilde{k}^\alpha [(1-\alpha)\theta + \alpha(1-\tau)] - (1-\tau)\delta \tilde{k} + \theta\} \right) \\
&= -\frac{(1-\tau)(\rho')^2 \tilde{k} f''}{D\theta^2 \beta \rho} [(1-\alpha-\alpha\beta)A \tilde{k}^\alpha + \beta\delta \tilde{k} + 1] \\
&> 0.
\end{aligned}$$

### 7.3 Proof of Proposition 5

We show that under Assumptions 2 and 3,

$$\frac{\rho'(\tilde{c})\tilde{k}}{\rho(\tilde{c})} > -\frac{\alpha\beta}{(1-\alpha)\delta} \tag{25}$$

holds for any pair of parameters, and hence  $g_A$  must be positive due to (14a) and  $\beta < 1 - \alpha$ .

Under Assumption 3, we have

$$\frac{\rho'(\tilde{c})\tilde{k}}{\rho(\tilde{c})} = -\frac{\beta\tilde{k}}{\tilde{c}+1} \tag{26}$$

and from Lemma 5 and Assumption 2,

$$\frac{\partial}{\partial A} \left( -\frac{\beta\tilde{k}}{\tilde{c}+1} \right) = -\frac{\beta\tilde{k}^{\alpha-1}\{[\alpha(1-\tau) + (1-\alpha)\theta]\beta\rho(\tilde{c})\tilde{k} + \alpha(1-\tau)\theta\}}{D\theta(\tilde{c}+1)^2} < 0. \tag{27}$$

Next we show that

$$\lim_{A \rightarrow \infty} \left( -\frac{\beta \tilde{k}}{\tilde{c} + 1} \right) = -\frac{\alpha \beta}{(1 - \alpha) \delta}. \quad (28)$$

One can easily verify that  $\lim_{A \rightarrow \infty} \tilde{k} = \lim_{A \rightarrow \infty} \tilde{c} = \infty$ . This implies  $\lim_{A \rightarrow \infty} A \hat{f}'(\tilde{k}) = \delta$ , because  $\lim_{c \rightarrow \infty} \rho(c) = 0$  and  $\rho(c) = (1 - \tau)[A \hat{f}'(k) - \delta]$  holds at any steady state. Then (28) holds since

$$\frac{\tilde{k}}{\tilde{c} + 1} = \frac{1}{(1 - \alpha) A \hat{f}'(\tilde{k}) / \alpha + (1 - \tau)[A \hat{f}'(\tilde{k}) - \delta] / \theta + 1 / \tilde{k}}.$$

From (26)–(28), we may conclude that (25) holds for any pair of parameters.

## 7.4 Proof of Proposition 7

It is apparent from (20b) that irrespective of the sign of  $\rho'$ , labor migration of from rural to urban areas raises the steady state welfare of rich households.

For that of poor households, we have

$$\begin{aligned} \frac{\partial \tilde{I}^*}{\partial \theta} &= \left[ w' + \frac{\tau \rho}{(1 - \theta)(1 - \tau)} \right] \frac{\partial \tilde{k}}{\partial \theta} + \frac{\tau \rho' \tilde{k}}{(1 - \theta)(1 - \tau)} \cdot \frac{\partial \tilde{c}}{\partial \theta} + \frac{\tau \rho \tilde{k}}{(1 - \theta)^2 (1 - \tau)} \\ &= \frac{1}{D \theta^2} \left( -\tilde{k} f'' \rho \rho' \tilde{k} + \frac{\tau \rho' \tilde{k}}{1 - \theta} \rho \tilde{k} f'' \right) + \frac{\tau \rho \tilde{k}}{(1 - \theta)^2 (1 - \tau)} \left[ \frac{(1 - \theta) \rho \rho'}{D \theta^2} + 1 \right] \\ &= \frac{\rho \tilde{k}}{(1 - \theta)^2 (1 - \tau) D} \left\{ -\frac{\tilde{k} f'' \rho' (1 - \theta - \tau)^2}{\theta^2} + \tau \left[ \frac{\rho \rho'}{\theta^2} - (1 - \tau) f'' \right] \right\}, \end{aligned}$$

which is negative for sufficiently small  $\tau$ , or  $\theta$  due to (23) under DMI.

Let  $\tau = 1 - \theta$ . Then,

$$D = \frac{\rho \rho'}{\theta} - \theta f'',$$

and

$$\frac{\partial \tilde{I}^*}{\partial \theta} = \frac{\rho \tilde{k}}{(1 - \theta) \theta^2 D} \left( \frac{\rho \rho'}{\theta} - \theta^2 f'' \right),$$

which is positive when  $\theta$  is sufficiently close to 1 since  $D > 0$ .

Under CMI, we have  $\partial \tilde{k} / \partial \theta = 0$  and  $\rho' = 0$ , and therefore

$$\frac{\partial \tilde{I}^*}{\partial \theta} = \frac{\tau \rho \tilde{k}}{(1 - \theta)^2 (1 - \tau)},$$

the sign of which corresponds to that of  $\tau$ .