***A Supply Chain Model with Reverse Information Exchange*: Complete Appendix**

Step 1: Defining the Restricted Process

We begin with focusing on a restricted version of the complete CTMC described above. The restricted version observes the process only over states for which $N \le S_m$. We note that whenever the restricted process is in states $\{i, S_m\}$, $0 \le i \le \Delta$, the next transition due to an external or retailer demand arrival can only result in either the restricted process reverting to the same state $\{i, S_m\}$ or to the state $\{\Delta, S_m\}$. When a retailer arrival occurs in state $\{i, S_m\}$, the unrestricted process will see a transition to $\{\Delta, S_m + \Delta - (i-1)\}$ which will ensure that the first state entered in the restricted process will be $\{\Delta, S_m\}$. When an external arrival occurs in state $\{i, S_m\}$, the unrestricted process will see a transition to $\{i, S_m + 1\}$. Irrespective of the specific sample path which the unrestricted process follows, it is certain that the first state it enters in the restricted state space will either be $\{i, S_m\}$ or $\{\Delta, S_m\}$ depending on whether any retailer arrival occurs between two consecutive visits to the states with $N = S_m$. Note that the restricted process is also a CTMC. Its state-transition-rate diagram is shown in the Figure 4.

Note that all the transition rates except those out of states $\{i, S_m\}$ remain the same as in the unrestricted process. In the following we define the transition rates for the following transitions: $\{i, S_m \mid i < \Delta\} \to \{i, S_m\}$ and $\{i, S_m \mid i < \Delta\} \to \{\Delta, S_m\}$. Let us first focus on the former transition. Define:

f_{ij}: Starting in a state $\{i, j \mid i < \Delta, j \ge S_m\}$ and conditioned on the next transition being due to external arrival, the probability that the unrestricted process will return to state $\{i, j\}$ for the first time before it reaches state $\{\Delta, S_m\}$.

Thus, f_{ij} is the probability that starting in $\{i, j \mid i < \Delta, j \ge S_m\}$, there is no retailer arrival until the process is back in the same state (any retailer arrival will result in $IP = S_u$ and therefore the process will first reach the state $\{\Delta, S_m\}$). Using standard conditioning arguments, we can write the recursion:

$$f_{ij} = q\sum_{k=0}^{\infty}(pf_{i.j+1})^k$$

where, $q = \left(\dfrac{\mu}{\lambda_e + \lambda_r + \mu}\right)$ represents the probability that once in state $\{i, j+1\}$, a service transition will bring the process back to $\{i, j\}$ and $p = \left(\dfrac{\lambda_e}{\lambda_e + \lambda_r + \mu}\right)$ represents the probability that once in state

$\{i, j+1\}$, an external arrival will take the process forward. The recursion above states that once in state $\{i, j+1\}$, the only way to go back to state $\{i, j\}$ is to either go through a service completion or if an external arrival occurs, then the process must come back to $\{i, j+1\}$ and then go through a service completion.

Next, from the structure of unrestricted Markov chain, it is clear that $f_{ij} = f_{i.j+1}$. Another way to see this is to simply note that the above recursive equation stands for all $j \geq S_m$ and $f_{i.j} = f_{i.j+1} \forall j \geq S_m$ will provide one valid solution to this system of recursive equations. Now, we can simplify the equation as: $f_{ij} = \dfrac{q}{1 - p f_{ij}}$ and, therefore, $f_{ij} = \dfrac{1 - \sqrt{1 - 4pq}}{2p}$. As this probability is the same for all $i < \Delta, j \geq S_m$, we will simply call it f.

Finally, to complete the characterization of the restricted Markov chain, the transition rate attached to $\{i, S_m\} \rightarrow \{i, S_m\}$ for $0 \leq i < \Delta$ will be $\lambda_e f$ and the rate for $\{i, S_m\} \rightarrow \{\Delta, S_m\}$ will be $\lambda_r + \lambda_e(1 - f)$. The rate for $\{\Delta, S_m\} \rightarrow \{\Delta, S_m\}$ will be $\lambda_e + \lambda_r$. We will normalize these rates by dividing them by μ. Thus, all the backward transitions with rate μ will be considered as having rate 1; transition rates $\lambda_e f$ are changed to $\rho_e f$; $\lambda_r + \lambda_e(1 - f)$ to $\rho_r + \rho_e(1 - f)$ and $\lambda_e + \lambda_r$ to $\rho_e + \rho_r$.

Step 2: Steady-State Probabilities of the Restricted Process

To be precise, we should define the steady-state probabilities (say, π'_{ij}) of the restricted process as separate from the steady-state probabilities (π_{ij}) of the unrestricted process. However, we will be only developing equations expressing π'_{ij} as a multiple of $\pi'_{\Delta.0}$ which would also hold for probabilities π_{ij}. To avoid clutter, we will work with notation π_{ij} even while addressing the restricted process.

First, focus on the states where $\omega = \Delta$. The balance equations are:

$$\pi_{\Delta.j} = (1 + \rho)\pi_{\Delta.j-1} - \rho_e \pi_{\Delta.j-2} \quad \text{for } 2 \leq j \leq S_m,$$

$$\pi_{\Delta 1} = \rho \pi_{\Delta 0}.$$

We note that the second-order homogeneous difference equation $y_{k+2} - (1 + \rho) y_{k+1} + \rho_e y_k = 0$ has the general solution $y_k = c_1 z_1^k + c_2 z_2^k$, where z_1, z_2 are the two roots of the characteristic equation

$z^2-(1+\rho)z+\rho_e=0$, and the two constants c_1, c_2 are determined by the two boundary conditions: $y_1=\rho\pi_{\Delta 0}; y_0=\pi_{\Delta 0}$. This gives:

$$z_1=\frac{(1+\rho)+\sqrt{(1+\rho)^2-4\rho_e}}{2}; z_2=\frac{(1+\rho)-\sqrt{(1+\rho)^2-4\rho_e}}{2};$$

$$c_1=\frac{\rho-z_2}{z_1-z_2}\pi_{\Delta 0}; c_2=\frac{z_1-\rho}{z_1-z_2}\pi_{\Delta 0},$$

and

$$\pi_{\Delta.j}=c_1 z_1^j+c_2 z_2^j \qquad \text{for } 0\le j\le S_m. \tag{A1}$$

We now introduce some simple properties of the roots z_1, z_2:

(a) $z_1+z_2=1+\rho$;

(b) $z_1 z_2=\rho_e$;

(c) $z_2=\rho_e f$.

To see (c), recall, that $f=\dfrac{1-\sqrt{1-4pq}}{2p}$ where $p=\left(\dfrac{\lambda_e}{\lambda_e+\lambda_r+\mu}\right)$ and $q=\left(\dfrac{\mu}{\lambda_e+\lambda_r+\mu}\right)$. Simple algebra shows that $z_2=\rho_e f$.

Next, let us focus on the states with $0<\omega<\Delta$. A similar approach can be taken by treating the balance equations as second-order inhomogeneous difference equations and then using boundary conditions to solve for unknown constants. However, the following approach simplifies the analysis. To analyze the states with $\omega=i$, we begin with writing the balance equation for the state (i, S_m):

$$(1+\rho)\pi_{i.S_m}=\rho_e\pi_{i.S_m-1}+\rho_e f\pi_{i.S_m}.$$

This gives:

$$\pi_{i.S_m}=z_2\pi_{i.S_m-1}.$$

Now, the balance equations for the states with $\omega=i$ can be written as:

$$\pi_{ij}=(1+\rho)\pi_{i.j-1}-\rho_e\pi_{i.j-2}-\rho_r\pi_{i+1,j-1} \text{ for } 2\le j\le S_m,$$

$$\pi_{i.1}=\rho\pi_{i.0}-\rho_r\pi_{i+1.0}.$$

Starting with $j=S_m$ and going backwards, for $0<i<\Delta$, we have:

$$\pi_{i.j}=z_2\pi_{i.j-1}+\rho_r\sum_{k=j}^{S_m-1}\frac{1}{z_1^{k-j+1}}\pi_{i-1.k} \text{ for } 1\le j\le S_m, \tag{A2}$$

$$\pi_{i.0} = \frac{\rho_r}{\rho - z_2} \sum_{k=0}^{S_m-1} \frac{1}{z_1^k} \pi_{i-1.k} \,. \tag{A3}$$

The last part of this step is to analyze states with $i = 0$. The approach is essentially the same as before. We first get:

$$\pi_{0.S_m} = \frac{\rho}{z_1} \pi_{0.S_m-1} \,.$$

Next, we write the balance equations for a group of states $\{(0, j) \mid 0 \le j \le S_m - 1\}$ as:

$$\pi_{0.S_m} = \rho\pi_{0.S_m-1} - \rho_r \sum_{k=0}^{S_m-1} \pi_{1.k} \,.$$

From the above two equations we get:

$$\pi_{0.S_m} = \frac{\rho_r}{(z_1 - 1)} \sum_{k=0}^{S_m-1} \pi_{1.k} \tag{A4}$$

Finally, writing the balance equations for the groups of states $\{(0,k) \mid 0 \le k \le j\}$, $0 \le j \le S_m - 1$ gives:

$$\pi_{0.j} = \frac{1}{\rho} \pi_{0.j+1} + \frac{\rho_r}{\rho} \sum_{k=0}^{j} \pi_{1.k} \qquad \text{for } 0 \le j \le S_m - 1 \tag{A5}$$

Step 3: Steady-State Probabilities for States with $\omega < \Delta; N > S_m$

Note that the balance equations for states with $N = S_m$ in the restricted process are:

$$(1+\rho)\pi_{iS_m} = \rho_e \pi_{iS_m-1} + \rho_e f \pi_{iS_m} \quad 0 < i < \Delta;$$

$$(1+\rho)\pi_{iS_m} = \rho \pi_{iS_m-1} + \rho_e f \pi_{iS_m} \quad i = 0.$$

In the unrestricted process, the balance equation for the same states can be written as:

$$(1+\rho)\pi_{iS_m} = \rho_e \pi_{iS_m-1} + \pi_{iS_m+1} \;\; \mathit{for} \;\; 0 < i < \Delta,$$

$$(1+\rho)\pi_{iS_m} = \rho \pi_{iS_m-1} + \pi_{iS_m+1} \;\; \mathit{for} \;\; i = 0.$$

The solution, which satisfies the above equations, is:

$$\pi_{iS_m+1} = \rho_e f \;\; \pi_{iS_m} \;\; \mathit{for} \;\; 0 \le i < \Delta \,.$$

Next, let us write the balance equations for the rest of the states, $N \ge S_m + 1$:

$$\pi_{ij} = (1+\rho)\pi_{i.j+1} - \rho_e \pi_{i.j-2} \;\; \text{for} \;\; 0 \le i < \Delta \,; \;\; j \ge S_m + 2 \,.$$

As in step 2, the general solution for this difference equation is $c_1 z_1^j + c_2 z_2^j$. Using boundary conditions gives:

$$c_1 z_1 + c_2 z_2 = \rho_e f \pi_{i.S_m},$$

$$c_1 + c_2 = \pi_{i.S_m}$$

and as we know that:

$$\rho_e f = z_2,$$

we get: $c_1 = 0; c_2 = \pi_{i.S_m}$. Finally,

$$\pi_{ij} = \pi_{i.S_m} z_2^{j-S_m} \quad 0 \le i < \Delta; j \ge S_m. \tag{A6}$$

Step 4: Steady-State Probabilities for States with $\omega = \Delta; N > S_m$

To determine the probability $\pi_{\Delta.j}$, $j \ge S_m + 1$, we will consider the balance equations for the aggregate state $\{(k,l) \mid k < \Delta, l \ge 0\} \bigcup \{(k,l) \mid k = \Delta, l \le j-1\}$. These equations can be written as:

$$\pi_{\Delta,j} = \rho\pi_{\Delta.j-1} + \rho_r \left(\sum_{k=0}^{\Delta-1} \sum_{l=S_m+[(j-S_m)-(\Delta-k+1)]^+}^{\infty} \pi_{k,l} \right) \quad j \ge S_m + 1.$$

Note that all the probabilities on the right hand side of the above expression can be expressed as multiples of $\pi_{\Delta.0}$. From (A6) and the result (c) from the proof of Theorem 2 below, we have:

$$\pi_{\Delta,j} = \rho\pi_{\Delta.j-1} + (\rho - z_2)\sum_{k=0}^{\Delta-1} \pi_{k.S_m} z_2^{[j-S_m-1-\Delta+k]^+} \quad j \ge S_m + 1 \tag{A7}$$

For states with $N \ge S_m + \Delta + 2$ this can be written as:

$$\pi_{\Delta,S_m+\Delta+1+j} = \rho^j \pi_{\Delta.S_m+\Delta+1} + \left(\rho^j - z_2^j\right) \sum_{k=0}^{\Delta-1} z_2^{k+1} \pi_{k,S_m} \quad j \ge 1$$

Proof of Theorem 1:

We note that (A1) does not depend on any artificial transition rates defined for the restricted process and therefore it is true for the unrestricted CTMC. Now (A2) through (A5) crucially depend on the artificial transitions from states $(i, S_m); 0 \le i < \Delta$ to themselves with rate $\rho_e f$. This leads to:

$$(1+\rho)\pi_{i.S_m} = \rho_e \pi_{i.S_m-1} + \rho_e f \pi_{i.S_m} \text{ for } 0 < i < \Delta$$

and

$$(1+\rho)\pi_{i.S_m} = \rho\pi_{i.S_m-1} + \rho_e f \pi_{i.S_m} \text{ for } i = 0.$$

In the unrestricted process, the balance equations for $(i, S_m); 0 \le i < \Delta$ can be written as:

$$(1+\rho)\pi_{i.S_m} = \rho_e \pi_{i.S_m-1} + \pi_{i.S_m+1} \text{ for } 0 < i < \Delta,$$

and

$$(1+\rho)\pi_{i.S_m} = \rho\pi_{i.S_m-1} + \pi_{i.S_m+1} \text{for } i = 0\,.$$

These two sets of equations are equivalent due to (A6) which gives:

$$\pi_{iS_m+1} = \rho_e f\ \pi_{iS_m}\quad for\ 0 \le i < \Delta\,.$$

Thus, (A2) through (A6) are mutually consistent and together satisfy the balance equations of unrestricted CTMC.

Finally, (A7) is again derived with no reference to the artificial transition rates of the restricted CTMC. To check that this holds in the restricted CTMC, note that the balance equation for $\{\Delta, S_m\}$ in the restricted CTMC is consistent with its balance equation in the unrestricted CTMC. That is, equations:

$$(1+\rho)\pi_{\Delta.S_m} = \rho_e \pi_{\Delta.S_m-1} + \rho\pi_{\Delta.S_m} + (\rho - z_2)\sum_{i=0}^{\Delta-1}\pi_{i.S_m}\ ,$$

and,

$$(1+\rho)\pi_{\Delta.S_m} = \rho_e\pi_{\Delta.S_m-1} + \pi_{\Delta.S_m+1}$$

are equivalent because from (A7):

$$\pi_{\Delta.S_m+1} = \rho\pi_{\Delta.S_m} + (\rho - z_2)\sum_{i=0}^{\Delta-1}\pi_{i.S_m}\ . \qquad \square$$

Proof of Theorem 2:

We will first establish several relationships that are required to prove the main assertion in the following:

(a) $\sum_{j=0}^{n}\pi_{1,j} = \sum_{i=1}^{\Delta}\pi_{i,n+1} - \rho_e\sum_{i=1}^{\Delta}\pi_{i,n}$ for $0 \le n \le S_m - 1$.

This can be seen directly from writing the balance equations for the set of states $\{(i,j)\,|\,1 \le i \le \Delta,\ 0 \le j \le n\}$.

(b) $\rho_r\sum_{j=0}^{S_m-1}\pi_{m,j} = (\rho - z_2)\sum_{i=0}^{m-1}\pi_{i,S_m}$, for $1 \le m \le \Delta$.

This can be seen directly from writing the balance equations for the set of states $\{(i,j)\,|\,m \le i \le \Delta,\ 0 \le j \le S_m\}$ in the restricted process.

(c) $\rho - z_2 = \dfrac{\rho_r}{(1 - z_2)}$.

To see this, note that

$$(\rho - z_2)(1 - z_2) = \{z_2^2 - (1+\rho)z_2 + \rho_e\} + \rho_r = \rho_r\,.$$

(d) $(1-\rho)\sum_{i=0}^{\Delta-1}\sum_{j=S_m}^{\infty}\pi_{i,j} = \frac{(1-\rho)}{(1-z_2)}\sum_{i=0}^{\Delta-1}\pi_{i,S_m}$.

This is simply due to $\pi_{ij} = \pi_{i.S_m} z_2^{j-S_m} \quad 0 \le i < \Delta;\ j \ge S_m$ as in Step 3 above.

(e) $(1-\rho)\sum_{j=S_m}^{\infty}\pi_{\Delta,j} = \pi_{\Delta,S_m} + (\rho - z_2)\sum_{i=0}^{\Delta-1}(\Delta - i)\pi_{i,S_m} + \frac{(\rho - z_2)}{(1-z_2)}\sum_{i=0}^{\Delta-1}\pi_{i,S_m}$.

To see this, from Step 4 we have:

$$\pi_{\Delta,j} = \rho\pi_{\Delta.j-1} + \rho_r\left(\sum_{k=0}^{\Delta-1}\ \sum_{l=S_m+[(j-S_m)-(\Delta-k+1)]^+}^{\infty}\pi_{k,l}\right) \quad j \ge S_m + 1.$$

Summing up the set of equations and employing:

$$\pi_{ij} = \pi_{i.S_m} z_2^{j-S_m} \quad 0 \le i < \Delta;\ j \ge S_m\,,$$

we obtain:

$$(1-\rho)\sum_{j=S_{m+1}}^{\infty}\pi_{\Delta,j} = \rho\,\pi_{\Delta,S_m} + \frac{\rho_r}{(1-z_2)}\sum_{i=0}^{\Delta-1}\{(\Delta - i) + \sum_{k=0}^{\infty} z_2^k\}\pi_{i,S_m}\ .$$

Adding $(1-\rho)\pi_{\Delta,S_m}$ on both sides and using (c), we get (e).

Now, we begin the last step in the proof by writing the balance equations for the set of states $\{(0, j) \mid 0 \le j \le n\}$ for all $0 \le n \le S_m - 1$ and employing (a) as follows:

$$\pi_{0,n+1} = \rho\,\pi_{0,n} - \rho_r\sum_{j=0}^{n}\pi_{1,j} = \rho\,\pi_{0,n} - \rho_r\left\{\sum_{i=1}^{\Delta}\pi_{i,n+1} - \rho_e\sum_{i=1}^{\Delta}\pi_{i,n}\right\} \text{ for } 0 \le n \le S_m - 1,$$

Summing up over n yields:

$$(1-\rho)\sum_{j=1}^{S_m-1}\pi_{0,j} + \pi_{0,S_m} = \rho\pi_{0,0} + \rho_e\sum_{i=1}^{\Delta}\pi_{i,0} - (1-\rho_e)\sum_{j=1}^{S_m-1}\sum_{i=1}^{\Delta}\pi_{i,j} - \sum_{i=1}^{\Delta}\pi_{i,S_m}$$

$$= \{-(1-\rho)+1\}\pi_{0,0} + \{-(1-\rho)+(1-\rho_r)\}\sum_{i=1}^{\Delta}\pi_{i,0} - \{(1-\rho)+\rho_r\}\sum_{j=1}^{S_m-1}\sum_{i=1}^{\Delta}\pi_{i,j} - \sum_{i=1}^{\Delta}\pi_{i,S_m}$$

[*Note:* This note provides a formula to express $\sum_{j=0}^{S_m-1}\pi_{o.j}$ in terms of $\pi_{i.S_m}, 0 \le i \le \Delta$. This was referred to in Section 5.

$$(1-\rho)\sum_{j=0}^{S_m-1}\pi_{0.j} = (1-\rho) - (1-\rho_e)\sum_{i=1}^{\Delta}\sum_{j=0}^{S_m-1}\pi_{i.j} - \sum_{i=0}^{\Delta}\pi_{i.S_m} = (1-\rho) - \frac{(1-\rho_e)}{(1-z_2)}\sum_{i=0}^{\Delta-1}(\Delta - i)\pi_{i.S_m} - \sum_{i=0}^{\Delta}\pi_{i.S_m}$$

or:

$$\sum_{j=0}^{S_m-1}\pi_{0.j} = 1-\sum_{i=0}^{\Delta}\left\{\frac{(1-\rho_e)(\Delta-i)}{(1-z_2)(1-\rho)}+\frac{1}{(1-\rho)}\right\}\pi_{i.S_m}\ ]$$

Some rearranging and changing the order of summation and employing (b), we have:

$$(1-\rho)\sum_{i=0}^{\Delta}\sum_{j=0}^{S_m-1}\pi_{i,j} = \pi_{0,0}+(1-\rho_r)\sum_{i=1}^{\Delta}\pi_{i,0}-\rho_r\sum_{j=1}^{S_m-1}\sum_{i=1}^{\Delta}\pi_{i,j}-\sum_{i=1}^{\Delta}\pi_{i,S_m}-\pi_{0,S_m}$$

$$=\sum_{i=0}^{\Delta}\pi_{i,0}-\rho_r\sum_{i=1}^{\Delta}\sum_{j=0}^{S_m-1}\pi_{i,j}-\sum_{i=0}^{\Delta}\pi_{i,S_m}\ .$$

$$=\sum_{i=0}^{\Delta}\pi_{i,0}-(\rho-z_2)\sum_{i=0}^{\Delta-1}(\Delta-i)\pi_{i,S_m}-\sum_{i=0}^{\Delta}\pi_{i,S_m}$$

Adding the above to those obtained in (d) and (e), we get:

$$(1-\rho)\sum_{i=0}^{\Delta}\sum_{j=0}^{S_m-1}\pi_{i,j}+(1-\rho)\sum_{i=0}^{\Delta-1}\sum_{j=S_m}^{\infty}\pi_{i,j}+(1-\rho)\sum_{j=S_m}^{\infty}\pi_{\Delta,j}$$

$$=\sum_{i=0}^{\Delta}\pi_{i,0}-(\rho-z_2)\sum_{i=0}^{\Delta-1}(\Delta-i)\pi_{i,S_m}-\sum_{i=0}^{\Delta}\pi_{i,S_m}+\frac{(1-\rho)}{(1-z_2)}\sum_{i=0}^{\Delta-1}\pi_{i,S_m}+\pi_{\Delta,S_m}+(\rho-z_2)\sum_{i=0}^{\Delta-1}(\Delta-i)\pi_{i,S_m}+\frac{(\rho-z_2)}{(1-z_2)}\sum_{i=0}^{\Delta-1}\pi_{i,S_m}$$

$$=\sum_{i=0}^{\Delta}\pi_{i,0}-\sum_{i=0}^{\Delta}\pi_{i,S_m}+\frac{(1-\rho)}{(1-z_2)}\sum_{i=0}^{\Delta-1}\pi_{i,S_m}+\pi_{\Delta,S_m}+\frac{(\rho-z_2)}{(1-z_2)}\sum_{i=0}^{\Delta-1}\pi_{i,S_m}$$

$$\Rightarrow (1-\rho)\sum_{i=0}^{\Delta}\sum_{j=0}^{\infty}\pi_{i.j}=\sum_{i=0}^{\Delta}\pi_{i.0}-\sum_{i=0}^{\Delta}\pi_{i.S_m}+\sum_{i=0}^{\Delta}\pi_{i.S_m}\ \Rightarrow \sum_{i=0}^{\Delta}\pi_{i.0}=(1-\rho)\ .\ \square$$

Proof of Theorem 3:

We first prove the convexity of $C[i,j,k]$ for $j \ge S_m$:

Recall that,

$$C[i,j,k]=(h+b)*E[Erlang(\lambda_r,u)-T-Erlang(\mu,v)]^{+}-b*\left[\frac{u}{\lambda_r}-T-\frac{v}{\mu}\right]$$

$$=h*\left[\frac{u}{\lambda_r}-T-\frac{v}{\mu}\right]+(h+b)*E[T+Erlang(\mu,v)-Erlang(\lambda_r,u)]^{+}$$

where $u=S_l+i+k-1$ and $v=j-S_m+k$ and $Erlang(\alpha,w)$ has a density $f(\alpha,w,t)$ given by:

$$f(\alpha,w,t)=\alpha\frac{(\alpha t)^{w-1}}{(w-1)!}e^{-\alpha t}\ \ for\ t\ge 0.$$

Therefore,

$$C[i,j,k]=$$

$$h\left(\frac{i+S_l+k-1}{\lambda_r}-T-\frac{j-S_m+k}{\mu}\right)+(h+b)\int_{y=0}^{\infty}\int_{x=0}^{y+T}(y+T-x)\lambda_r\,p(i+S_l+k-2,\lambda_r x)\mu p(j-S_m+k,\mu y)\,dx\,dy$$

where $p(a,b)=\frac{b^a}{a!}e^{-b}$ is the Poisson mass function and $P(a;b)=\sum_{j=a}^{\infty}p(j,b)$. Let $g(S_l)$ be defined as follows:

$$g(S_l)=\int_{x=0}^{y+T}(y+T-x)\lambda_r\,p(i+S_l+k-2;\lambda_r x)\,dx$$

$$=(y+T)\,P[i+S_l+k-1;\lambda_r(y+T)]-\frac{i+S_l+k-1}{\lambda_r}P[i+S_l+k;\lambda_r(y+T)]$$

Denote $\delta_{.}$ as the first order difference operator; that is:

$$\delta_{S_l}\,g(S_l)=g(S_l)-g(S_l-1)$$

$$=-(y+T)\,p(i+S_l+k-2;\lambda_r(y+T))+\frac{i+S_l+k}{\lambda_r}p(i+S_l+k-1;\lambda_r(y+T))$$

$$-\frac{1}{\lambda_r}P[S_l+i+k-1;\lambda_r(y+T)]$$

$$=-\frac{1}{\lambda_r}P[S_l+i+k-1;\lambda_r(y+T)].$$

Therefore:

$$\delta_{S_l}C[i,j,k]=\frac{h}{\lambda_r}-(b+h)\int_0^{\infty}\frac{\mu}{\lambda_r}P[S_l+i+k-1;\lambda_r(y+T)]\;p(j-S_m+k-1;\mu y)\,dy,$$

and,

$$\delta_{S_l}^2C[i,j,k]=(b+h)\int_0^{\infty}\frac{\mu}{\lambda_r}p[S_l+i+k-1;\lambda_r(y+T)]\;p(j-S_m+k-1;\mu y)\,dy\quad>0.$$

Thus $C[i,j,k]$ is convex in S_l. In a similar fashion we can show that $C[0,0,1]$ is convex in S_l. Now, recall that:

$$C_{S_l.\Delta}=\lambda_r\left[\sum_{i=0}^{\Delta}\sum_{j=0}^{\infty}\pi_{ij}C[i,j]\right]=\lambda_r\left[\sum_{j=0}^{S_m-1}\pi_{0j}C[0,0,1]+\sum_{i=0}^{\Delta}\sum_{j=S_m}^{\infty}\pi_{ij}\sum_{k=1}^{\Delta-i+1}C[i,j,k]\right]$$

This implies that $\delta_{S_l}^2C_{S_l.\Delta}>0$. □

Proof of Theorem 4:

We will use the symbol $\pi_{\Sigma.j}$ to denote $\Pr[N^{RI} = j] = \sum_{i=0}^{\Delta} \pi_{i.j} = \pi_{\Sigma.j}$. In *NI*, $\Pr[N^{NI} = j] = (1-\rho)\rho^{j}$.

We need to show that :

$$\Pr[N^{RI} \le j] \le \Pr[N^{NI} \le j].$$

From Theorem 2, we have $\pi_{\Sigma.0} = 1-\rho$. Using the aggregation/disaggregation approach to analyze Markov chains (see Schweitzer 1991), we can write:

$$\pi_{\Sigma.1} = \left(\rho_e + \rho_r \frac{\pi_{0.0}}{\pi_{\Sigma.0}}\right)\pi_{\Sigma.0} < \rho\pi_{\Sigma.0}.$$

Alternatively, this can be seen by simply writing the balance equation for the group of states $\{\pi_{i.0} \mid 0 \le i \le \Delta\}$. Similarly, we can write $\pi_{\Sigma.j} < \rho\pi_{\Sigma.j-1} \quad for \ \ j \le S_m$ by writing the balance equations for the groups of states $\{\pi_{i.k} \mid 0 \le i \le \Delta; k \le j\} \quad j \le S_m - 1$. Next, writing the balance equation for the group of states $\{\pi_{i.j} \mid 0 \le i \le \Delta;\ j \le S_m\}$, we get $\pi_{\Sigma.S_m+1} = \rho\pi_{\Sigma.S_m}$. Therefore:

$$\Pr[N^{RI} = j] \le \Pr[N^{NI} = j]$$

and

$$\Pr[N^{RI} \le j] \le \Pr[N^{NI} \le j] \quad for \ \ j \le S_m + 1.$$

Next, consider the balance equation for the group of states $\{\pi_{i.j} \mid 0 \le i \le \Delta;\ j \le S_m\}$, which gives:

$$\pi_{\Sigma.S_m+2} = \rho\pi_{\Sigma.S_m+1} + \rho_r \sum_{i=0}^{\Delta} \pi_{i.S_m} > \rho\pi_{\Sigma.S_m+1}.$$

Similarly,

$$\pi_{\Sigma.S_m+j} > \rho\pi_{\Sigma.S_m+j-1} \quad for \ \ j \ge 2.$$

Therefore,

$$\pi_{\Sigma.S_m+j} > \rho^{j-k}\pi_{\Sigma.S_m+k} \quad for \ \ j > k \ge 1.$$

We will prove the rest by contradiction. Assume there exists:

$$j^* = Min\{k \mid \sum_{j=0}^{k} \pi_{\Sigma.j} \ge \Pr[N^{NI} \le k];\ k \ge S_m + 2\}.$$

From the definition of j^*,:

$$\Pr[N^{RI} = j^*] = \pi_{\Sigma.j^*} > \Pr[N^{NI} = j^*] = (1-\rho)\rho^{j^*}.$$

Now,

$$\underset{j\to\infty}{Lim}\Pr[N^{RI} \le j] = \sum_{j=0}^{j^*}\pi_{\Sigma.j} + \sum_{j=j^*+1}^{\infty}\pi_{\Sigma.j}$$

$$> \Pr[N^{NI} \le j^*] + \sum_{j=j^*+1}^{\infty}\rho^{j-j^*}\Pr[N^{NI} = j^*] = (1-\rho)\frac{(1-\rho^{j^*+1})}{(1-\rho)} + (1-\rho)\rho^{j^*}\frac{\rho}{(1-\rho)} = 1 .$$

This is impossible. Therefore no such j^* exists and $\Pr[N^{RI} \le j] \le \Pr[N^{NI} \le j]$ for all $j \ge 0$. □

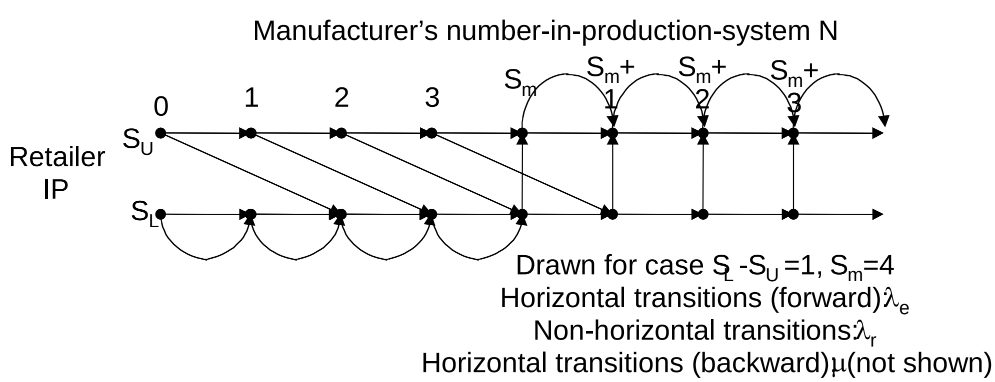
Proof of Proposition 5:

Recall from the Theorem 4 that $\Pr[N^{RI} = j] \le \Pr[N^{NI} = j] \quad for \;\; j \le S_m$. Thus:

$$E[S_m - N]^{+RI} = \sum_{j=0}^{S_m-1}(S_m - j)\Pr[N^{RI} = j] \le \sum_{j=0}^{S_m-1}(S_m - j)\Pr[N^{NI} = j] = E[S_m - N]^{+NI} . \quad □$$

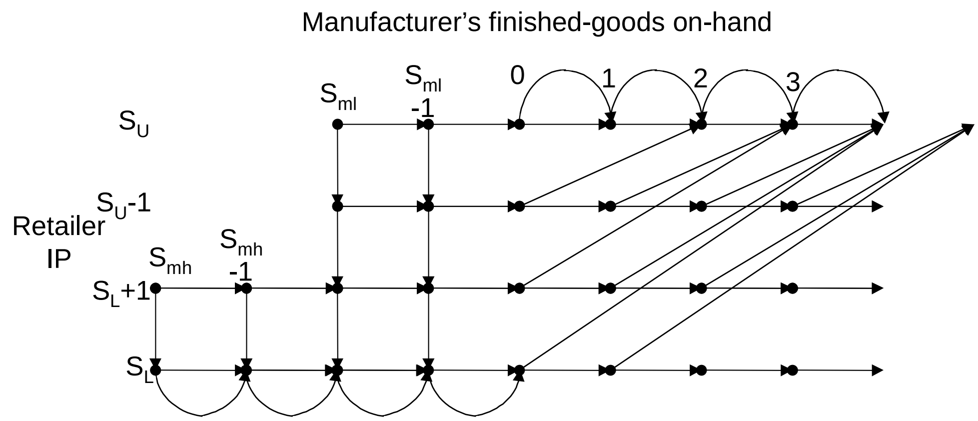
Extensions of the Basic Model: Case of $S_l = S_u + 1$ (Section 2: Retailer's Policy)

The following figure sketches out the CTMC for this case. We can follow essentially the same analytical steps as we employed in the case of $S_u > S_l$ to develop the stead-state probabilities for this Markov chain. A restricted Markov chain here will consist of all the states with $N \le S_m + 1$. The finite state restricted chain can be easily solved. The cost calculation also follows the same steps as presented in Section 4.



Extensions of the Basic Model: Manufacturer's Two-Level Policy (Section 5.3)

The following CTMC presents the case when the manufacturer is using the policy described in Section 5.3. Note that we have changed the state description from number in production system to finished goods on-hand inventory. The manufacturer knows the retailer's inventory position. If it is greater than S_{ip}, then the manufacturer stops production when on-hand inventory reaches S_{ml} otherwise the production continues until on-hand inventory reaches S_{mh}. A restricted process can be created exactly in the same way as described in the paper. The solution of the restricted chain has only minor differences and the rest of the analysis follows exactly the same steps.



Drawn for case S_U-S_L=3, S_{mh}=4, S_{ml}=2, S_{ip}=S_L+1
Horizontal transitions (forward): λ_e
Non-horizontal transitions: λ_r
Horizontal transitions (backward): μ(not shown)