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# Economic control and inspection policies for high-speed unreliable production systems

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In this paper, we consider a high-speed production process, which produces defects at a known rate while in control. When the process goes out of control, it produces defects at a higher rate. In this study, we revisit the role of the distribution of the process in-control time when managing such systems. Specifically, we focus on two management schemes, a control policy and an inspection policy. In the control policy, when the number of defects produced reaches a threshold, the process is stopped and inspected. In contrast, in the inspection policy, the process is stopped and inspected periodically. We derive the operating characteristics of the system and devise schemes for finding the optimal policy parameters for each policy. We also investigate the behavior of the optimal policy parameters, compare the performances of the control and inspection policies and identify the environments in which each of these policies out performs the other one using a numerical experiment.

## 1. Introduction

In this paper, we consider the economic design of control and inspection policies in unreliable high-speed/high-volume production processes. High-speed production systems are common in practice and can be found in many industries. Examples range from traditional industries such as potato grading (Noordam *et al.*, 2000), aluminum beer can manufacture (Gold, 1993) and metal forming (Schoch, 1994), to high-tech industries such as semiconductor wafer production (Schonecker *et al.*, 2002) and image printing (King and West, 1995). In such environments, it is usual to have integrated devices, which perform automated measurements of the output of the process that are necessary for next stage of production. The measurements are made possible through the use of optical sensors (Gold, 1993; King and West, 1995) such as laser or infrared devices resulting in no sampling cost and thus, full sampling of all units produced.

Many manufacturing processes are unreliable in nature since machines wear down after a period of intensive use and this results in an increased defective rate and thus, excessive salvage costs for undetected defectives. Therefore, it is necessary to have a mechanism, which ensures the timely halting of the process, identifies an assignable cause and restores the process to its original state. In this study, we design and analyze policies to detect machine breakdowns so as to minimize the average operating cost rate involved in high-speed production processes. Our policies are economic in nature, since various operational costs will be explicitly incorporated.

The economic design of control charts is an intensively studied topic dating back to Duncan (1956) who considered the $\bar{X}$-chart. Later Goel and Wu (1973) and Chiu (1974) proposed a Cumulative SUM (CUSUM) policy for controlling the quality of production systems. They assumed a continuous process, which stays in-control according to an exponential distribution. While in control, the process produces defects with a known mean and variance. The mean defect rate shifts to a higher rate after machine breaks down (process goes out of control). Simpson and Keats (1996) provided an optimization scheme and performed a sensitivity study for an economic control model using a CUSUM policy. Q1

Lorenzen and Vance (1986) presented a unified approach that can systematically determine the economic design of various control charts. McWilliams (1989) extended the analysis to the case where the in-control time follows a Weibull distribution enabling one to consider systems where in-control times have increasing and decreasing hazard rates. He re-examined Lorenzen and Vance (1986) and found that the economic design of the standard control charts is quite insensitive to the shape of the distribution of in-control times. Furthermore, McWilliams (1996) discussed the relationship between it—earlier control models and the unified Lorenzen–Vance model, and derived an approximation scheme in order to find the optimal control parameters. More recently, Linderman and Love (2001) extended the Lorenzen–Vance model to develop the economic design of Multivariate Exponentially Weighted Moving Average (MEWMA) control charts.

In this paper, we consider a high-speed production system, which produces defects at a known rate while in control. When the process goes out of control, it produces defects at a higher rate. For any given distribution of the in-control time, we revisit the role of information, defined as the coefficient of variation for the in-control time, when managing such systems. Specifically, we focus on two schemes for managing such systems: a control policy in line with that of CUSUM charts and an inspection policy. In the control policy, each unit produced is sampled and defective units are identified. When the number of defects in the process reaches a limit, the process is stopped and the machine is inspected for an assignable cause. In contrast, in the inspection policy, units produced are not sampled; however, at predetermined time points, the process is stopped and the machine itself is inspected. Since the distribution of the in-control times is not necessarily exponential, then the inspection intervals may not be of equal lengths. An interesting question to be investigated is to compare these two different schemes (control policies in comparison to inspection policies) and identify the type of environment in which one of these policies performs better than the other one.

In our model, we assume that units are produced with a known defect rate. Now, in the high-speed production environment, our underlying process converges to a Brownian motion (or Wiener process) in which its drift increases when the process goes out of control. This is known as the change-point problem (Carlstein, *et al.*, 1994) which has attracted significant attention from mathematicians and statisticians. However, previous research has focused on minimizing the average run length (ARL), which ignores information on the various costs, and often yields suboptimal solutions. Our control policy is designed as an upper bound (on the number of defective units) parallel to the process mean drift in the in-control state. The distance of the upper bound to the mean drift is optimally determined, by incorporating the costs associated with machine repairs, unit salvage, and false alarms. Among related works, Ye (1990) is noteworthy since considered a scenario that is somewhat similar to our setting and focused on the optimal timing for machine replacement when maintenance and operational costs evolve according to a Wiener process. These cost parameters define the state of the machines which deteriorate over time.

As previously mentioned we will also propose a time-based inspection policy, in which the machine will be inspected periodically (not necessarily at equal time intervals). There exist many studies that consider time-based inspection policies to manage unreliable systems. Badia *et al.* (2001) considered a model in which the failure times are assumed to be either exponential or Pareto-type and analyzed a periodic inspection policy in the presence of inspection errors. Lee and Rosenblatt (1987) studied the joint determination of the optimal production cycle and inspection schedules in unreliable production systems. They showed that, for an exponentially-distributed time-to-shift the inter-inspection times should be equally spaced. Moinzadeh and Klastorin (1995) considered a system that produces no defective items when in control but produces defective items of a given defective rate when out of control. Assuming that the in-control time is exponentially distributed, they introduce the idea of locating a buffer after the production system that is used to prevent defective units produced between inspections being transfered to the next stage of production. More recently Berk and Moinzadeh (2000) studied a time-based maintenance policy for $M$ machines whose performance deteriorated with use. Similarly, a set of optimal inspection times can be found by balancing all costs involved.

Our inspection policy relaxes the assumption of an exponential time-to-failure distribution. The idea is to compare its performance with that of control polices for different in-control time distributions. In our numerical experiments, we have chosen top one an Erlang distribution for the in-control times and this enables us to consider various distributions ranging from deterministic (complete memory) to exponential (memoryless). Intuition suggests that in one limit when the distribution is exponential, the control policy outperforms the inspection policy since the inspection policy ignores the characteristics of the underlying process and the exponential in-control time distribution lacks a memory. At the other end of the spectrum with a complete memory (deterministic), the inspection policy is expected to be superior. Indeed, this is confirmed in our study. In addition, we show that inspection policies dominate the control policies when there is a significant memory on the distribution of the in-control times whereas the control policy performs better as the level of memory is decreased.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries regarding the production process and the notation. Section 3 studies the control policy whereas the inspection policy is examined in Section 4. Section 5 offers conclusions and directions for future research.

## 2. Preliminaries

In this section, we first introduce the model and its assumptions and define the relevant notations. Then we discuss some of characteristics of the manufacturing process under study.

### 2.1. *The Model, notations and assumptions*

Consider an unreliable production system that produces units at a rate $\gamma$, which is typically high in high-speed/high-volume production settings. The process is in-control producing defective units at a rate $p_1$ for a random amount of time, $\tau$, before it goes out of control (breaks down). When out of control, the process produces defective items at a rate of $p_2$ ($p_2 > p_1$). For a given policy, when certain policy conditions are met, the machine is stopped and inspected to find possible assignable causes of breakdown (being out of control). If there is a breakdown then the machine is repaired and restored to in-control condition; otherwise, a "false alarm" happens. We denote $C_{\mathrm{mr}}$ as the machine repair/restoration cost and $C_{\mathrm{fa}}$ ($C_{\mathrm{fa}} < C_{\mathrm{mr}}$) as the cost of a false alarm. Furthermore, we assume that the defective units produced are discarded at a salvage cost, $C_{\mathrm{ur}}$, per unit. Finally, we assume that all the units produced are sampled and tested (100% sampling) implying that the unit sampling cost is zero. This assumption is reasonable for processes in which sampling and testing is peformed using optical sensors that measure a specified quality parameter. In such settings the sampling cost is negligible since the measurement is part of production.

We now summarize the relevant parameters used in our modeling studies.

***Parameters:***

$\gamma$ = production rate;
$\tau_0$ = mean in-control time;
$k$ = variance parameter for in-control time;
$p_1$ = in-control defective rate;
$p_2$ = out-of-control defective rate ($p_2 > p_1$);
$C_{\mathrm{fa}}$ = false alarm cost;
$C_{\mathrm{ur}}$ = unit product salvage cost;
$C_{\mathrm{mr}}$ = machine repair cost;

***Decision variables:***

$L$ = control limit;
$\Delta_n$ = inspection times ($n = 1, 2, \dots$).

### 2.2. *Process characteristics*

To model the manufacturing process, let us first define a random walk:

$$S_N = x_1 + x_2 + \cdots + x_N, \qquad (1)$$

and $S_0 = 0$. Since each unit produced is sampled then $S_i$ denotes the number of defective units when a total of $i$ units are produced. The Bernoulli process to describe the production of defectives is defined by $P(x_i = 0) = 1 - p$ and $P(x_i = 1) = p$, where $p = p_1$ or $p_2$ for in-control or out-of-control states, respectively.

Since units are produced at a rate of $\gamma$ then the number of units produced in $t$ time units is equal to $N = \gamma t$. In the limit where $\gamma$ is large, the random walk $S_N$ can be approximated by a continuous process, which follows a Brownian motion $S(t)$ with probability density:

$$P(S(t) = y) = P(y; t) = \frac{1}{(2\pi\sigma^2 t)^{1/2}} \exp\left(-\frac{(y-\mu t)^2}{2\sigma^2 t}\right), \qquad (2)$$

where $\mu = p\gamma$ is the drift velocity and $\sigma^2$ is the variance parameter and is equal to $\sigma_t^2 = 2\sigma^2 \;\; t\sigma^2 = p(1-p)\gamma/2$. This is a direct result of the central limit theorem (Bhattacharya and Waymire, 1990).

To analyze the control policy in the next section, we need the probability density function for the first passage time to a barrier $z$, which is given by (Bhattacharya and Waymire, 1990).

$$f_{\sigma,\mu;z}(t) = \frac{|z|}{(2\pi\sigma^2)^{1/2} t^{3/2}} \exp\left(-\frac{(z-\mu t)^2}{2\sigma^2 t}\right). \qquad (3)$$

When the drift is less than or equal to the barrier, that is, $z/\mu \le 0$, the probability of reaching the barrier is less than one, and the expected passage time is also infinite. If $z/\mu > 0$, then the expected first passage time to a barrier $z$ is given by:

$$E(t) = \int_0^\infty t f_{\sigma,\mu;z}(t)\mathrm{d}t = \int_0^\infty t \frac{|z|}{(2\pi\sigma^2)^{1/2} t^{3/2}} \exp\left(-\frac{(z-\mu t)^2}{2\sigma^2 t}\right)\mathrm{d}t = \frac{z}{\mu}, \qquad (4)$$

which is independent of the "diffusion constant" $\sigma$.

## 3. The control policy

### 3.1. *Policy definition*

We now propose a control policy to manage the system described earlier. The policy is based on tracking the process $S(t)$ defined in the previous section and can be stated as follows.

When,

$$S(t) \ge \mu t + L, \qquad (5)$$

the machine is stopped and inspected. If an assignable cause is detected, the process is repaired and restored to its original condition; otherwise, a false alarm has occurred. Here $L$ is the control limit that will be determined so that the average cost/time will be minimized and $\mu = p_1\gamma$ is the drift velocity when the process is in control. Obviously, the choice of the slope in Equation (5) is heuristic as it is not determined optimally. We define the production cycle as the time between two consecutive machine repairs. Note that during a cycle, there may be a number of false alarms. When the process is restarted after a false alarm, $S(t)$ is *reset* to be $\mu t$ where $t$ is the time (elapsed from the beginning of a cycle) which the false alarm occurred.

In order to find the average total cost rate, we need to evaluate the average total cost in a cycle and the average cycle time. For ease of exposition, we will work with the process $S(t) - \mu t$. In this reference frame, when the process is in control, the drift velocity is zero, but the variance parameter is $\sigma^2 = p_1(1-p_1)\gamma/2$. When the process is out of control, the relative drift velocity is $\delta\mu = (p_2 - p_1)\gamma$.

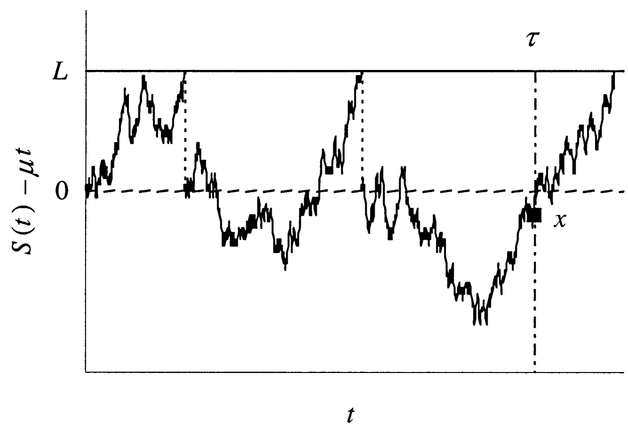


**Fig. 1.** Typical scenario for a process.

Figure 1 depicts the policy setting in which we assume that the process is out of control when $t > \tau$. Here $\tau$ is the machine breakdown time and $x$ is the location where the process $S(t) - \mu t$ ends up at time $t = \tau$.

### 3.2. *Policy analysis*

As mentioned before, to analyze this policy and find the optimal control bound $L$, we need to figure out the expected number of false alarms and the expected cycle time. A false alarm occurs when the process $S(t)$ reaches the control bound at any time before $t = \tau$, where $\tau$ is the machine breakdown time. Let us write $f(t) = f_{\sigma,\mu=0;L}(t)$, and denote $P_0(\tau)$ as the probability that there is no false alarm during a cycle, then:

$$P_0(\tau) = \int_\tau^\infty f(t)\mathrm{d}t. \tag{6}$$

Equation (6) states that the first passage time happens when $t \geq \tau$, therefore no false alarm is triggered. The probability there are $n$ false alarms during a cycle, $P_n(\tau)$, can be calculated recursively as:

$$P_n(\tau) = \int_0^\tau f(t)P_{n-1}(\tau - t)\mathrm{d}t = \int_0^\tau f(\tau - t)P_{n-1}(t)\mathrm{d}t. \tag{7}$$

The above recursion results from the fact that, the *first* false alarm occurs at a time $t$ and there are $(n-1)$ false alarms for the remaining $\tau - t$.

The expected number of false alarms in a cycle, is:

$$En(\tau) = \sum_{n=0}^{\infty} nP_n(\tau). \tag{8}$$

$En(\tau)$ can also be obtained by solving an integral equation. Multiplying both sides of Equation (7) by $n$ and summing over $n$ gives:

$$En(\tau) = \int_0^\tau f(\tau - t)En(t)\mathrm{d}t + P_0(\tau). \tag{9}$$

This is a Volterra equation of the second kind (Jerri, 1999) and is typically solved by using the Laplace transform. We first find the Laplace transform for $f(t)$, explicitly (Abramowitz and Stegun, 1972):

$$\mathcal{L} \circ f(\lambda) = \int_0^\infty f(t)\exp(-\lambda t)\mathrm{d}t = \exp(-\sqrt{2\lambda}L/\sigma). \tag{10}$$

Applying Laplace convolution (Faltung) theorem (Gradshteyn *et al.*, 1998), the solution to Equation (9) can be expressed in terms of inverse Laplace transform as:

$$En(\tau) = \frac{1}{2\pi i}\int_{\zeta - i\infty}^{\zeta + i\infty} \frac{1}{\lambda}\frac{\exp(-\sqrt{2\lambda}L/\sigma)}{1 - \exp(-\sqrt{2\lambda}L/\sigma)}\exp(\lambda\tau)\mathrm{d}\lambda, \tag{11}$$

where $\zeta$ is a real constant that exceeds the real part of all the singularities of the integrand.

Next, we find the probability density that the process ends at $x$ at time $\tau$ (see Fig. 1), which we denote as $\Pr(x;\tau)$. This will be used when calculating the elapsed time between the machine breaking down ($t = \tau$) and time at which the breakdown is detected. First, the probability density that the process reaches $x$ at time $\tau$ *without* hitting the control limit can be written as:

$$P_0(x;\tau) = \frac{1}{(2\pi\sigma^2\tau)^{1/2}}\exp\left(-\frac{x^2}{2\sigma^2\tau}\right) - \frac{1}{(2\pi\sigma^2\tau)^{1/2}}\exp\left(-\frac{(2L-x)^2}{2\sigma^2\tau}\right), \tag{12}$$

where $x \leq L$. In Appendix 1 we have provided a proof for more generic situations, although Equation (12) can be derived from the reflection principle (Bhattacharya and Waymire, 1990). Similarly, the probability that the process reaches $x$ at time $\tau$ *and* there are $n$ false alarms before $\tau$, $P_n(x;\tau)$, can be calculated recursively from:

$$P_n(x;\tau) = \int_0^\tau f(t)P_{n-1}(x;\tau - t)\mathrm{d}t, \tag{13}$$

where $1 \leq n < \infty$. Finally, $\Pr(x;\tau) = \sum_{n=0}^{\infty} P_n(x;\tau)$, satisfies:

$$\Pr(x;\tau) = \int_0^\tau f(\tau - t)\Pr(x;t)\mathrm{d}t + P_0(x;\tau). \tag{14}$$

Again, Equation (14) is a Volterra equation of the second kind whose solution can be explicitly expressed in terms of Laplace transforms, for $x \leq L$:

$$\Pr(x;\tau) = \frac{1}{2\pi i}\int_{\zeta - i\infty}^{\zeta + i\infty}\frac{1}{\sqrt{2\lambda}\sigma} \times \frac{\exp(-\sqrt{2\lambda}|x|/\sigma) - \exp(-\sqrt{2\lambda}(2L - x)/\sigma)}{1 - \exp(-\sqrt{2\lambda}L/\sigma)}\exp(\lambda\tau)\mathrm{d}\lambda. \tag{15}$$

For $t > \tau$, the process will be out of control and we can consider a new Brownian motion starting at time $\tau$ and from position $x$ with a drift velocity $\delta\mu$. From Equation

(4), we can write the expected time the process is in an out-of-control state during a cycle as:

$$T_{\text{ctl}}(\tau) - \tau = \frac{L - Ex(\tau)}{\delta\mu}, \tag{16}$$

where $T_{\text{ctl}}(\tau)$ is the expected duration of a production cycle and $Ex(\tau)$ is the expected value of the starting position for this new Brownian motion which, from Equation (15), can be expressed as;

$$\begin{aligned} Ex(\tau) &= \int_{-\infty}^{L} x \Pr(x; \tau)\mathrm{d}x \\ &= -L \cdot \frac{1}{2\pi i}\int_{\zeta - i\infty}^{\zeta + i\infty} \frac{1}{\lambda}\frac{\exp(-\sqrt{2\lambda}L/\sigma)}{1 - \exp(-\sqrt{2\lambda}L/\sigma)} \exp(\lambda\tau)\mathrm{d}\lambda \\ &= -L \times En(\tau). \end{aligned} \tag{17}$$

From Equation (17), we note that $Ex(\tau) + L \times En(\tau) = 0$. This is intuitive since the process (Brownian motion for $t < \tau$) has a zero drift velocity and in the absence of the control bound $L$ and resetting the process to $\mu t$ in the case of false alarms, the process should average to zero. The more false alarms (or the more resetting), the further away the process will end from the control limit.

### 3.3. *Optimal policy*

Now we are in a position to derive the optimal process control policy parameter $L$ which minimizes the average total cost rate defined as the ratio of the average total cost in a cycle to the average cycle time. The expected total number of defective units produced in a cycle is simply, $(En(\tau) + 1)L$. Given the unit false alarm cost $C_{\text{fa}}$, the unit product salvage cost $C_{\text{ur}}$, and machine repair cost $C_{\text{mr}}$, the average total cost during a cycle is:

$$TC_{\text{ctl}}(\tau) = En(\tau)C_{\text{fa}} + (En(\tau) + 1)LC_{\text{ur}} + C_{\text{mr}}. \tag{18}$$

For the convenience of future calculations, we express Equations (16) and (18) in terms of Laplace transforms. The Laplace transform for the expected cycle time is:

$$\mathcal{L} \circ T_{\text{ctl}}(\lambda) = \frac{1}{\lambda^2} + \frac{L}{\delta\mu}\frac{1}{\lambda}\frac{1}{1 - \exp(-\sqrt{2\lambda}L/\sigma)}. \tag{19}$$

Similarly, the expected total cost in a cycle is:

$$\begin{aligned} \mathcal{L} \circ TC_{\text{ctl}}(\lambda) = {}& \frac{1}{\lambda}\frac{\exp(-\sqrt{2\lambda}L/\sigma)}{1 - \exp(-\sqrt{2\lambda}L/\sigma)}C_{\text{fa}} \\ &+ \frac{1}{\lambda}\frac{L}{1 - \exp(-\sqrt{2\lambda}L/\sigma)}C_{\text{ur}} + \frac{C_{\text{mr}}}{\lambda}. \end{aligned} \tag{20}$$

The average total cost rate can be found from Equations (19) and (20). Before presenting our numerical experiment, we first study a special case where the machine breakdown time is exponential. In this case, the average total cost rate is given by the ratio of Equation (20) to Equation (19), with $\lambda$ replaced by $1/\tau_0$. The first-order condition of optimality with respect to $L$ gives:

$$\begin{aligned} &\left(\sqrt{\frac{2}{\tau_0}}\frac{L}{\sigma}\left(C_{\text{ur}} - \frac{C_{\text{mr}} - C_{\text{fa}}}{\delta\mu\tau_0}\right) + C_{\text{ur}}\right. \\ &\left. - \frac{C_{\text{mr}} - C_{\text{fa}}}{\delta\mu\tau_0} + \sqrt{\frac{2}{\tau_0}}\frac{C_{\text{fa}}}{\sigma}\right) \exp\left(-\sqrt{\frac{2}{\tau_0}}\frac{L}{\sigma}\right) = C_{\text{ur}} - \frac{C_{\text{mr}}}{\delta\mu\tau_0}. \end{aligned} \tag{21}$$

As shown in Appendix 2 the solution of above equation exists only when $\delta\mu\tau_0 C_{\text{ur}} > C_{\text{mr}}$. This states that there is no need to detect and repair machines in the out-of-control state if the average machine repair cost exceeds the average salvage cost of the defective units.

### 3.4. *Numerical results*

One focus of this paper is to investigate the impact of the distribution of the machine breakdown time $\tau$ on the control policy. For this, we numerically study a series of distributions for $\tau$:

$$\frac{1}{(k-1)!}\frac{k}{\tau_0}\left(\frac{k}{\tau_0}\tau\right)^{k-1} \exp\left(-\frac{k}{\tau_0}\tau\right), \tag{22}$$

where $k \geq 1$ is an integer. This is a series of Erlang distribution, $E_k$ which has a mean $\tau_0$ and a variance $\tau_0^2/k$. One advantage of the Erlang distribution is that the calculation of the expected (with respect to $\tau$) value of any function, say $g(\tau)$, can be reduced to taking derivatives of its corresponding Laplace transform, explicitly:

$$\frac{1}{(k-1)!}\left(\frac{k}{\tau_0}\right)^k (-1)^{k-1} \left.\frac{\partial}{\partial\lambda}\mathcal{L} \circ g(\lambda)\right|_{\lambda = k/\tau_0}. \tag{23}$$

In the case where $k = \infty$, that is, the deterministic machine breakdown time, we adopt an algorithm for inversion of the Laplace transform (Stehfest, 1970):

$$g(x) \approx \frac{\ln 2}{x}\sum_{i=1}^{N} V_i \mathcal{L} \circ g\left(\frac{\ln 2}{x} i\right), \tag{24}$$

where

$$V_i = (-1)^{\frac{N}{2} + i} \sum_{j = [(i+1)/2]}^{\min(i, N/2)} \frac{j^{N/2}(2j)!}{(N/2 - j)!\, j!\,(j-1)!\,(i-j)!\,(2j-i)!}. \tag{25}$$

In order to conduct numerical investigations, we have normalized the expected breakdown time and machine repair cost to unity, that is, $\tau_0 = 1$ and $C_{\text{mr}} = 1$. The defective rate in the in-control state is fixed at $p_1 = 0.2$. In addition, we set the product salvage cost per time unit, $\gamma C_{\text{ur}}$, to 10, and vary the production rate $\gamma$. A typical value for the production rate is $\gamma = 10\,000$, and for the unit salvage cost it is $C_{\text{ur}} = 0.001$. The typical values for the defective rate in the out-of-control state $p_2$ and the false alarm cost, $C_{\text{fa}}$,

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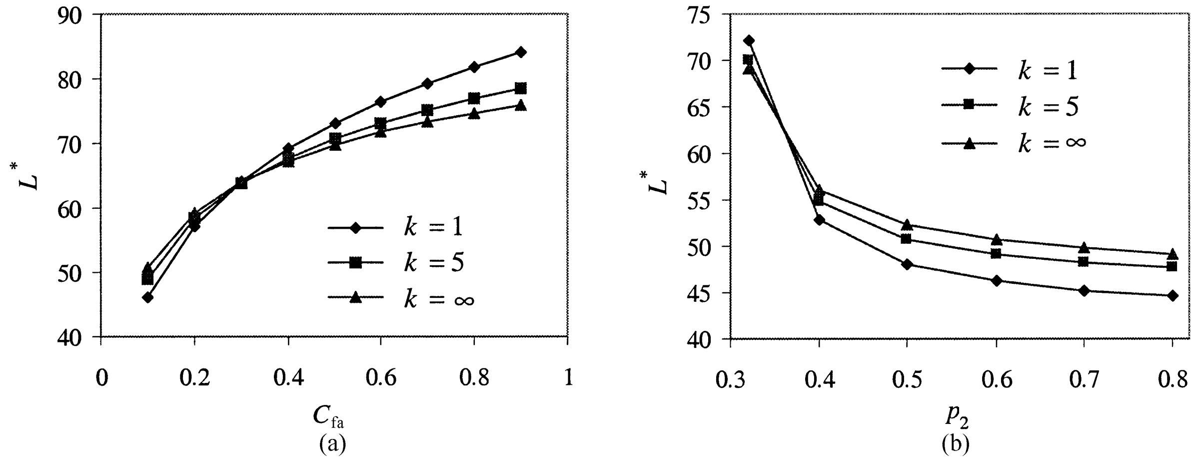



**Fig. 2.** (a) Optimal control limit plotted against the cost of a false alarm; and (b) the optimal control limit plotted against the out-of-control defective rate.

are set to be 0.6 and 0.1, respectively. In all the numerical experiments throughout the rest of this paper, we use these typical values defined above, unless stated otherwise. In the following, we will single out several important parameters and explore their impacts on the optimal control policy parameter, $L^*$.

In Fig. 2(a), we plot the optimal control limit as a function of the false alarm cost as the value of $k$ is varied. Recall that $k$ defines the degree of variability present in the distribution of in-control times (actually, the coefficient of variation of the in-control times is $k^{-1/2}$). In other words, $k$ determines the degree of memory of the distribution of the *in-control* times. The larger the value of $k$, the more memory is present on the distribution of the in-control times. When $k = 1$, the distribution of in-control times is memoryless (exponential). In contrast, $k = \infty$ presents the case where the distribution of the in-control times has a full memory (deterministic). It can be observed in Fig. 2(a), that the optimal control limit is decreasing in the cost of the false alarms. This is as expected, as a higher control limit should be set in order to avoid an increasing cost of the false alarms. We also notice that the $L^*$ increases more slowly with $C_{\text{fa}}$ when $k$ becomes larger; that is, for small values of $C_{\text{fa}}$ the optimal control limit is increasing in $k$ and as $C_{\text{fa}}$ becomes large, the reverse is true. This observation can be explained as follows: when the cost of a false alarm is small and the distribution of the in-control times has little memory (for instance when it is memoryless which corresponds to $k = 1$), the optimal policy should be to opt to reduce the associated costs of producing defects when out of control. This yields a lower optimal control limit (a tighter control policy), compared to the cases of larger values of $k$, which have more memory about the timing of the process breakdown. In contrast, when the cost of a false alarm is high, with less memory about the timing of the breakdown, the optimal policy should be to opt to reduce this cost by setting higher values of the control limit (a looser control policy).

Figure 2(b) depicts the behavior of the optimal control limit as the out-of-control defective rate $p_2$ and also $k$ are varied. As can be seen, the optimal control limit is decreasing in $p_2$. This is intuitive since a higher $p_2$ indicates that more defective units will be produced after the machine breaks down. Therefore, a lower control is set so that the out-of-control state can be detected earlier. Furthermore, as in Fig. 2(a), $L^*$ increases more slowly with $p_2$ when, $k$ becomes larger. This behavior can be explained along the same lines as described before; that is, when $p_2$ is small then, systems with less memory on the distribution of their in-control times opt to incur this cost rather than the cost of a false alarm by setting their control limit to a high value. When $p_2$ is large, the reverse holds this since systems with less memory adopt a tighter policy, which reduces their salvage costs at the expense of incurring a lower false alarm cost.

A more comprehensive depiction of the behavior of the optimal control limit as a function of the memory of the in-control time distribution (or $k$) appears in Fig. 3(a–c). As discussed earlier, depending on the magnitude of the cost of a false alarm $C_{\text{fa}}$, compared to the average of producing a defective/time when out of control, the optimal control limit can be strictly decreasing (when $C_{\text{fa}}$ is small), increasing (when $C_{\text{fa}}$ is large) or can be decreasing and then increasing (when $C_{\text{fa}}$ is medium). This behavior can be explained following the same line of reasoning given before. Furthermore, as can be seen from Fig. 3(a–c), the magnitude of change in the optimal control limit is small. Also, the optimal control bound seems to quickly converge to the corresponding value for the deterministic case represented by a dashed line.

It is useful to observe how the average cost of the optimal policy changes with process parameters. The average cost is plotted against the false alarm cost in Fig. 4(a) and out-of-control defective rate in Fig. 4(b). It is intuitive that the average cost increases monotonically with $C_{\text{fa}}$ and $p_2$.

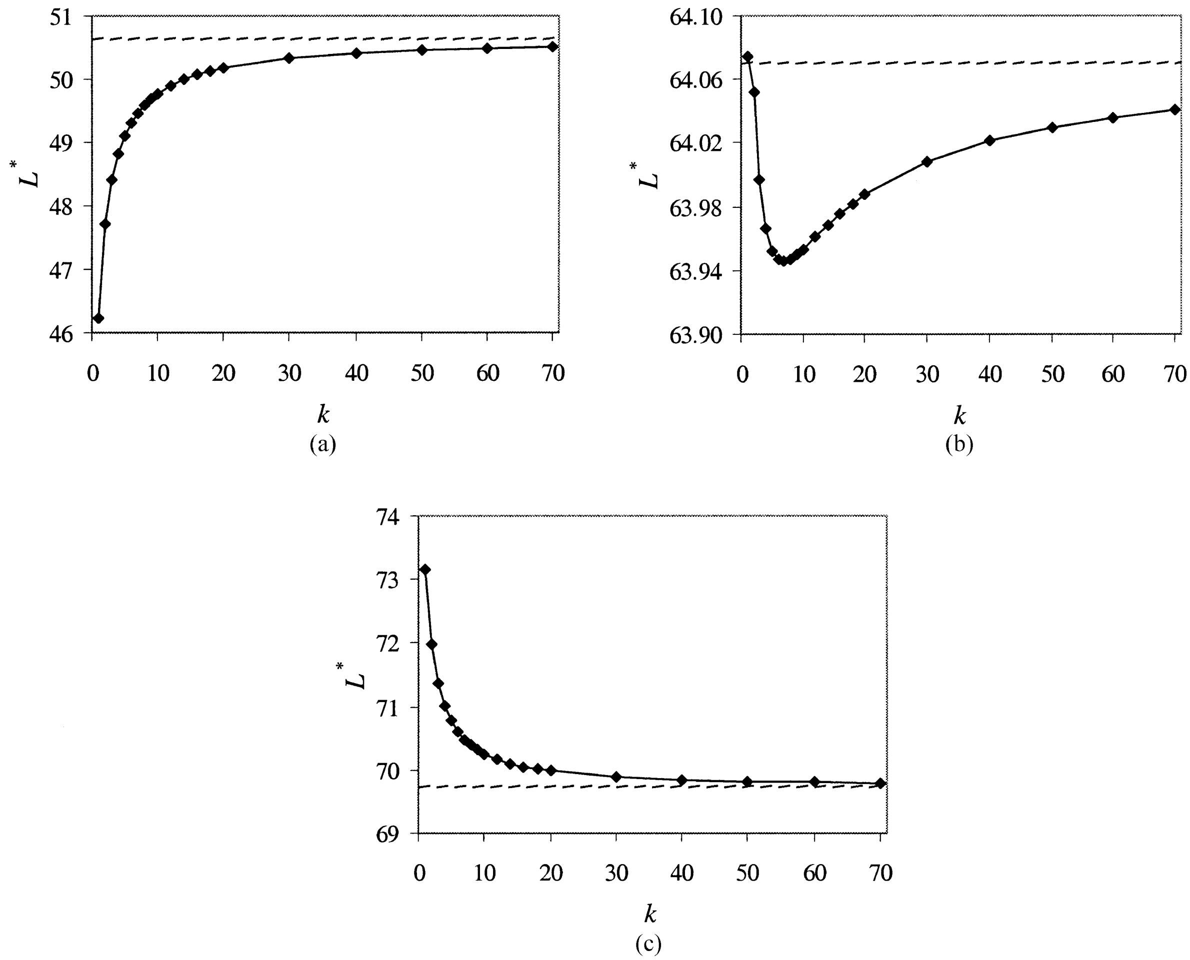



**Fig. 3.** Optimal control limit plotted as a function of $k$ for different $C_{\mathrm{fa}}$ values: (a) $c_{\mathrm{fa}} = 0.1$; (b) $C_{\mathrm{fa}} = 0.3$; and (c) $C_{\mathrm{fa}} = 0.5$.

As is shown in Fig. 4(b), a memory of the in-control time (described by $k$) does not seem to have a significant impact on the average cost when $p_2$ is increased. On the contrary, the average cost increases much faster with $C_{\mathrm{fa}}$ when the in-control time distribution has less memory. With high-production rates, the control limit is quickly reached once the process is out of control. Therefore, the expected cost of a false alarm dominates the salvage cost of the defective units when out of control, and is more sensitive to the in-control time distribution.

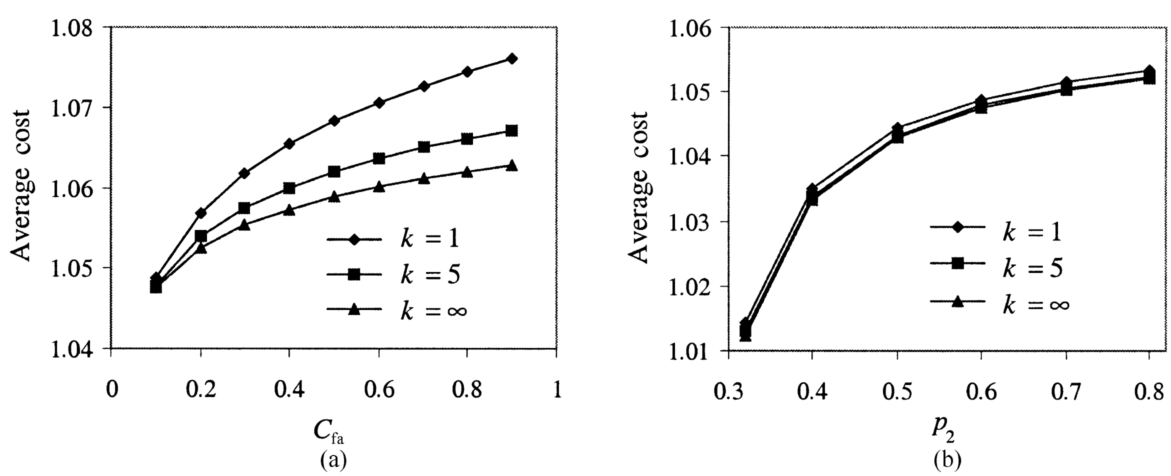



**Fig. 4.** (a) Average cost plotted against the cost of a false alarm; and (b) the average cost plotted against the out-of-control defective rate.

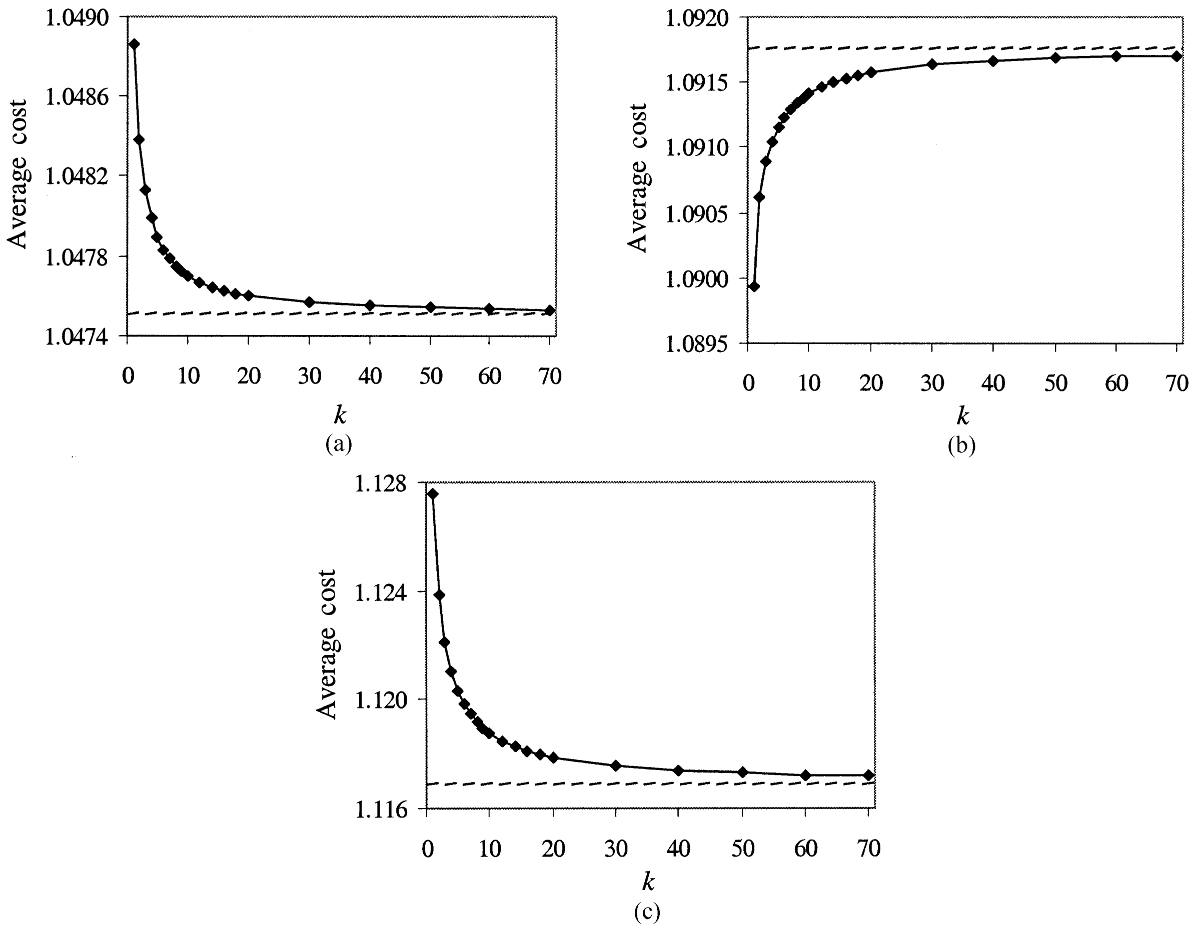



**Fig. 5.** Average cost plotted as a function of $k$ for different $C_{\mathrm{fa}}$ and $\gamma$ values: (a) $C_{\mathrm{fa}} = 0.1$; $\gamma = 10\,000$; (b) $C_{\mathrm{fa}} = 0.1$, $\gamma = 2000$. and $C_{\mathrm{fa}} = 0.05$, $\gamma = 2000$.

Figure 5(a–c) shows the effect of the false alarm cost and production rate on the average cost. It is observed that typically the average cost decreases as more information (or memory) on the in-control time (higher $k$) is available. However, with a slow machine and a low cost for a false alarm, less memory of the in-control times is preferred. Similarly, this can be explained by identifying when the expected false alarm cost dominates the salvage cost of defective units when out of control.

## 4. Inspection policy

In this section, we introduce a different class of policies to control such systems, namely, an inspection policy that is based on time and not the number of defective units produced. First, we will analyze this policy in this section. We then follow by presenting the behavior of the optimal policy parameter as the degree of memory present, characterized by the variability of the in-control time distribution, is varied in numerical examples. We then close this section by discussing the performance of inspection policies compared to the control policy studied in the previous section and identify the environments where one would dominate the other through a numerical experiment.

The analysis of this policy resembles that of the unified Lorenzen-Vance model. A comparison of model parameters used in our policy (M&T) and the unified Lorenzen-Vance model (L&V) is summarized in Table 1. In our policy, we assume that the sampling cost of the produced units is negligible since the measurement of units is a part of the production process, though this assumption can be easily relaxed.

**Table 1.** Parameters compared with the Lorenzen-Vance model parameters

| *Descriptions* | *L & V* | *M & T* |
|---|---|---|
| Quality cost/hour while in control | $C_0$ | Normalized to zero |
| Quality cost/hour while out of control | $C_1$ | $C_{\mathrm{ur}}$ (proportional to $C_1 - C_0$) |
| Cost per false alarm | $Y$ | $C_{\mathrm{fa}}$ |
| Cost to repair | $W$ | $C_{\mathrm{mr}}$ |
| Fixed cost per sample | $a$ | Assumed to be zero |
| Cost per unit sampled | $b$ | Assumed to be zero |

Note that there are a few key differences between the two models. First, in L&V, the sampling cost is positive. Therefore, the sample size and the control limits are the decision variables. In contrast, in our control policy, since the sampling cost is negligible, each unit produced is sampled. Moreover, in our inspection policy, the inspection is performed directly on the machine rather than the produced units. Second, L&V assumes an exponentially-distributed in-control time. Our model is more general since we consider an Erlang family, which enables us to study the impact of information, characterized by the coefficient of variation of the in-control times, on control/inspection decisions. In L&V, since the in-control times are exponentially distributed, the optimal inspection intervals are equally spaced. However, as our model allows for any arbitrary in-control time distribution, the optimal inspection intervals will not be equally spaced. Finally, L&V's numerical example is related to a foundry operation. The cost and other parameters are not directly applicable to our case of a high-speed/high-volume production system.

### 4.1. *Policy analysis*

The inspection policy is defined as follows. The machine is stopped and inspected at a set of pre-determined time points, $T_1, T_2, \ldots, T_n, \ldots$. If the machine is found to be out of control, it will be repaired and restored and a new cycle is started. Otherwise, the machine is put back to work, however, a false alarm cost is incurred.

To analyze the policy, we first define the inter-inspection time:

$$\Delta_n = T_n - T_{n-1}, \tag{26}$$

where $T_0 = 0$. and the unit step function:

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{27}$$

If the machine breaks down at time $t = \tau$, it is straightforward to explicitly express the cycle time using the unit step function:

$$T_{\text{inp}}(\tau) = \sum_{n=0}^{\infty} \Delta_{n+1} u(\tau - T_n). \tag{28}$$

Similarly, the total number of false alarms is:

$$\sum_{n=0}^{\infty} u(\tau - T_n) - 1. \tag{29}$$

This gives the total cost during a cycle as:

$$TC_{\text{inp}}(\tau) = \left( \sum_{n=0}^{\infty} u(\tau - T_n) - 1 \right) C_{\text{fa}} + \delta\mu \left( \sum_{n=0}^{\infty} \Delta_{n+1} u(\tau - T_n) - \tau \right) C_{\text{ur}} + C_{\text{mr}}. \tag{30}$$

For a given distribution of $\tau$, the expected cycle time is:

$$T_{\text{inp}} = \sum_{n=0}^{\infty} \Delta_{n+1} F(T_n), \tag{31}$$

where $F(\cdot)$ is the complementary cummulative distribution function of $\tau$. The expected total cost per cycle is:

$$TC_{\text{inp}} = \sum_{n=1}^{\infty} F(T_n) C_{\text{fa}} + \delta\mu \left( \sum_{n=0}^{\infty} \Delta_{n+1} F(T_n) - \tau_0 \right) C_{\text{ur}} + C_{\text{mr}}. \tag{32}$$

The expected average cost rate can be calculated as $AC_{\text{inp}} = TC_{\text{inp}}/T_{\text{inp}}$, which is a function of the inspection times $T_n$.

To derive the optimal set of inspection times, we examine the first-order condition:

$$\frac{\partial}{\partial T_n} AC_{\text{inp}} = 0, \tag{33}$$

which yields:

$$\frac{f(T_n)}{F(T_{n-1}) - F(T_n) - (T_{n+1} - T_n) f(T_n)} = \frac{\delta\mu C_{\text{ur}} - AC_{\text{inp}}}{C_{\text{fa}}} \equiv \frac{1}{\eta}. \tag{34}$$

Notice that the parameter $\eta$ is a constant *independent* of $n$. Its value can be obtained self-consistently using Equation (34) and the optimal values of $T_n$. This suggests an iterative way to solve the problem. More specifically, we can write:

$$\Delta_{n+1} = -\eta + \varphi(T_n, \Delta_n), \tag{35}$$

where $\varphi(T_n, \Delta_n)$ is defined as:

$$\varphi(T_n, \Delta_n) = \frac{F(T_n - \Delta_n) - F(T_n)}{f(T_n)}. \tag{36}$$

It is obvious that $\varphi(T_n, \Delta_n)$ is an increasing function of $\Delta_n$.

Similarly as in the previous section, we will now focus on the Erlang distribution ($E_k$) of $\tau$, given in Equation (22) as the choice for the in-control times. We will show that when in-control times are $E_k$, the optimal inter-inspection times will be non-increasing. To do so, we first start with the case where $k = 1$ (i.e., the exponential distribution). When $k = 1$, $\varphi(T_n, \Delta_n) = \tau_0(\exp(\Delta_n/\tau_0) - 1)$, which is independent of $T_n$. This reflects the memoryless property of the exponential distribution.

In Fig. 6, we have plotted Equation (35) for the exponentially-distributed machine breakdown time. Now we argue that the inter-inspection times should be equal; that is, $\Delta_n = \Delta^*$, where the value of $\Delta^*$ is indicated in Fig. 6. If $\Delta_1 < \Delta^*$, using Fig. 6, one can immediately observe that $\Delta_2 < \Delta_1$. Furthermore, $\Delta_n$ decreases monotonically with $n$ and becomes zero at some point resulting in an infinite cost for an expected false alarm. Therefore, this set of $\Delta_n$ (where $\Delta_1 < \Delta$) is excluded as a possible optimal solution. Also, since all the inspections combined together

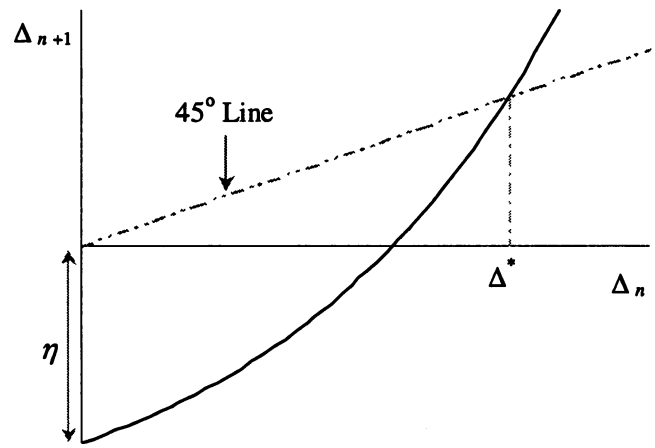



**Fig. 6.** Recursive relation for inter-inspection times ($k = 1$).

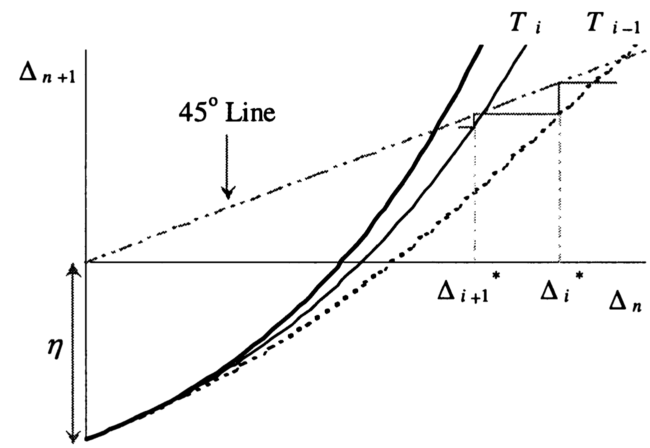



**Fig. 7.** Recursive relation for inspection times.

should cover the entire time axis, this scenario is not feasible since for a set of decreasing $\Delta_n$, $\sum_{n=1}^{\infty} \Delta_n \neq \infty$. Next, when $\Delta_1 > \Delta^*$, $\Delta_n$ will be monotonically increasing, according to Fig. 6. We have shown in Appendix 3 that this choice results in a higher average cost when compared to $\Delta_n = \Delta^*$. Therefore, we can conclude that $\Delta_n = \Delta^*$. This agrees with Lee and Rosenblatt (1987) who showed that, for an exponentially-distributed time-to-shift the time intervals between inspections should be equally spaced.

Now let us consider the case when $k \neq 1$ as shown in Fig. 7. As can be seen, all curves of $\varphi(T_n, \Delta_n)$ are dispersed. According to the property of an incomplete gamma function (Abramowitz and Stegun, 1972), for a fixed $\Delta_n$, $\varphi(T_n, \Delta_n)$ increases with $T_n$. The higher the value of the parameter $k$, the further apart these curves are separated. For a given $k$, we observe that:

$$\lim_{T_n \to \infty} \varphi(T_n, \Delta_n) = \frac{\tau_0}{k}\left(\exp\left(\frac{k}{\tau_0}\Delta_n\right) - 1\right), \qquad (37)$$

which is independent of $T_n$. From these properties, it is clear that the optimal inspection time intervals decrease with $n$; that is, $\Delta_{n+1}^* < \Delta_n^*$. As $n$ becomes larger, $\Delta_n^*$ converges to a non-zero value, determined by Equations (35) and (37). The higher is the value of $k$ then the faster $\Delta_n^*$ decreases with $n$.

### 4.2. *Numerical illustrations*

We now investigate the effectiveness of the inspection policy through a numerical experiment. In doing so, we first discuss the behavior of the optimal inspection intervals as the coefficient of variation ($k^{-1/2}$) is varied. Then, we compare the performance of the inspection policy with that of fixed inspection intervals. Finally, we close by discussing the performance of our inspection policy compared to the control policy studied in the previous section and identify the

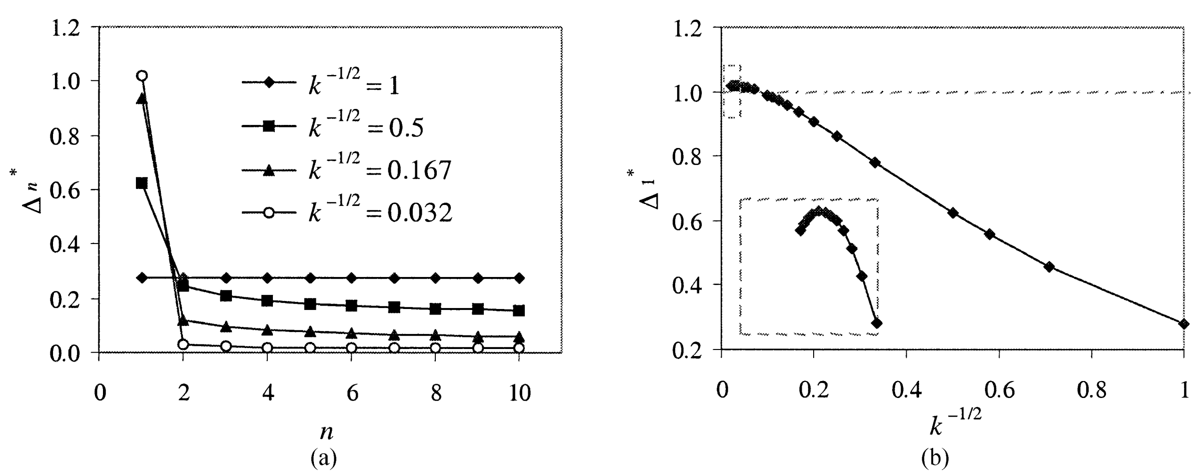



**Fig. 8.** (a) Optimal inspection intervals plotted against $n$ for different $k^{-1/2}$; and (b) optimal first inspection interval as a function of $k^{t1/2}$.

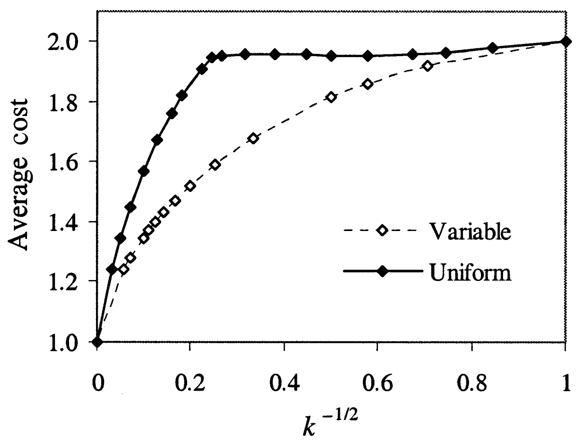



**Fig. 9.** Average costs as a function of the coefficient of variation $k^{-1/2}$.

environments where one would dominate the other through a numerical experiment.

In Fig. 8(a), the optimal inspection intervals $\Delta_n^*$ for $n = 1, 2, \ldots,$ are plotted for different values of $k^{t1/2}$. It can be observed that $\Delta_n^*$ drops fairly rapidly with $n$ and quickly converges to a non-zero value. Note that for lower values of $k^{-1/2}$, the first inspection will be close to the mean machine breakdown time ($\tau_0 = 1$), followed by very frequent inspections.

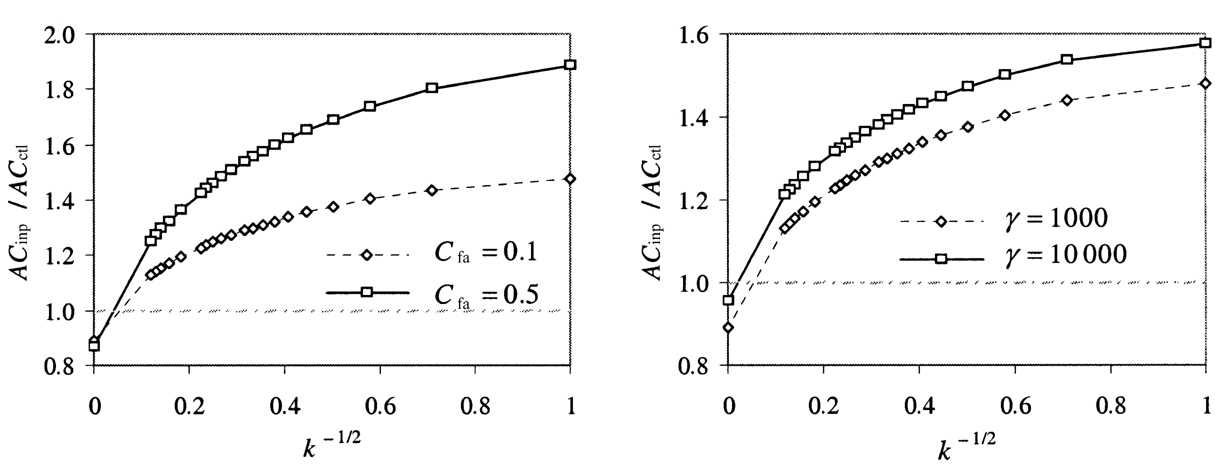
Figure 8(b) depicts the behavior of the optimal first inspection time, $\Delta_x^*$. As $k^{-1/2}$ decreases, it becomes more certain as to when the machine breaks down and $\Delta_x^*$ approaches the mean breakdown time $\tau_0 = 1$. It is interesting to observe (see the insert) that $\Delta_1^*$ will cross $\tau_0 = 1$ first and then come back. This is intuitive since when $k^{-1/2} = 0$ (i.e., the deterministic in-control times), $\Delta_1^* = \tau_0 + \varepsilon$, where $\varepsilon$ is infinitesimal indicating that the machine should be stopped right after it breaks down, so that a false alarm cost is not incurred.

A frequently used variation of the inspection policy proposed in this paper is that which sets the inspection intervals equally (we refer to this policy as a uniform inspection policy). Recall that this policy is optimal when the distribution of in-control times is either exponential (memoryless) or deterministic (complete memory). In Fig. 9, we study the performance of such a policy when compared to the one proposed in this paper which allows the inspection intervals to vary. Obviously, our inspection policy will not do worse than the uniform inspection policy. As a matter of fact, except in the two extreme cases, $k^{-1/2} = 1$ (exponential) and $k^{-1/2} = 0$ (deterministic), the non-uniform (variable) policy significantly outperforms the periodic one, especially for intermediate values of $k^{-1/2}$. This implies that when inspection is used to manage systems in which the in-control time distribution has some memory, inspection intervals should be decided by incorporating the information of the in-control time distribution. Complete ignorance of the memory present on the distribution of the in-control times and use of uniform inspection will result in an inferior performance.

Finally, we close by comparing the performance of our inspection policy to that of the control policy proposed earlier. As discussed earlier, the optimal policy should switch from the control policy to the inspection policy when there is more memory (a smaller coefficient of variation) about the in-control time.

In Fig. 10(a and b) we have plotted the ratio of average costs for inspection and control policies, $AC_{\text{inp}}/AC_{\text{ctl}}$. For the parameter values of our choice, it can be observed that the control policy outperforms ($AC_{\text{inp}}/AC_{\text{ctl}} > 1$) the inspection one for a wide range of values for the coefficient of variation. The inspection policy will outperform the control policy only when the coefficient of variation is very small.

As can be seen in Fig. 10(a), for a higher cost for a false alarm (for $\gamma = 1000$), the ratio $AC_{\text{inp}}/AC_{\text{ctl}}$ increases faster since there is less memory (or the coefficient of variation




(a) (b)

**Fig. 10.** Ratio of the average costs as a function of the coefficient of variation for: (a) different false alarm costs; and (b) different production rates.

is higher). This is intuitive since when facing a higher cost for a false alarm the control policy is more flexible in adjusting its control limit to a higher value in order to avoid triggering false alarms. For a fixed cost of false alarm ($C_{\mathrm{fa}} = 0.1$), as shown in Fig. 10(b), the control policy is more dominant for faster production systems, since the ratio $AC_{\mathrm{inp}}/AC_{\mathrm{ctl}}$ is higher. This is because a faster machine has a larger drift velocity after the machine breaks down and this allows the process to quickly reach the control limit.

It is worth noting that our control policy triggers an alarm based only on a single point above the threshold. However, most statistical process control software packages are implemented to combine other statistical criteria as suggested in the Western Electric Statistical Process Control Handbook (Anon, 1956) to the traditional control rules (i.e., control limits). We note that in settings such as ours, in the same spirit, a similar set of rules can be developed to complement our policy in practice.

## 5. Conclusions

In this paper, we have studied the economic design of control and inspection policies for high-speed unreliable production systems. We modeled the number of defective units produced as a Brownian motion. The mean drift of the process increases when the process is out of control. We have presented and analyzed a control policy that stops the process when the number of defective units exceeds a threshold. A time-based inspection policy that inspects the process periodically (not necessarily with equal time intervals) was also proposed and studied.

A main focus of this study was to examine the impact of in-control time distributions when managing such systems and the choice of policies (control or inspection) that should be employed in such environments. In doing so, we considered systems in which the in-control time distribution followed a series of Erlang distributions. First, we showed that when inspection is used to manage such systems, setting the inspection times equally which is widely practiced, may result in inferior performance in light of the presence of a memory on the distribution of the in-control times. Second, we showed that when managing such systems, the control policy seems to outperform the inspection policy in most settings. In fact, the inspection policy will only outperform the control policy when the coefficient of variation of the in-control time is small.

One possible extension to this work is to integrate both the control and inspection policies together. This approach makes use of information for both the underlying process and in-control time, and is expected to perform better. In this study, we have chosen the control limit for the control policy in a heuristic manner. Another possible extension is to consider a control policy in which the functional form of the control limit is determined optimally.

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## Appendices

### *Appendix 1*

The probability density of ending in $y$ $(<L)$ at time $t$ without touching the limit $L$ is:

$$P_0(y;t) = \frac{1}{(2\pi\sigma^2 t)^{1/2}}\left(\exp\left(-\frac{(y-\mu t)^2}{2\sigma^2 t}\right) - \exp\left(-\frac{2(L-y)\mu}{\sigma^2}\right)\exp\left(-\frac{(2L-y-\mu t)^2}{2\sigma^2 t}\right)\right). \tag{A1}$$

To show this, we first have the probability density of crossing the limit at least once:

$$P_c(y;t) = \int_0^t f_{\sigma,\mu;L}(\tau) P(y-L; t-\tau)\, d\tau.$$

Applying the Laplace transform:

$$\mathcal{L}\circ f_{\sigma,\mu;L}(\lambda) = \exp\left(\frac{L\mu}{\sigma^2}\right)\exp\left(-\frac{L}{\sigma}\sqrt{2\lambda+\frac{\mu^2}{\sigma^2}}\right);$$

$$\mathcal{L}\circ P(y-L;\lambda) = \frac{1}{\sqrt{4\lambda\sigma^2+2\mu^2}}\exp\left(\frac{(y-L)\mu}{\sigma^2}\right) \times \exp\left(-\frac{L-y}{\sigma}\sqrt{2\lambda+\frac{\mu^2}{\sigma^2}}\right).$$

Therefore,

$$\mathcal{L}\circ P_c(y;\lambda) = \frac{1}{\sqrt{4\lambda\sigma^2+2\mu^2}}\exp\left(\frac{y\mu}{\sigma^2}\right) \times \exp\left(-\frac{2L-y}{\sigma}\sqrt{2\lambda+\frac{\mu^2}{\sigma^2}}\right).$$

Laplace inverting the above expression and subtracting it from $P(y,t)$ gives Equation (A1).

### *Appendix 2*

Equation (21) can be written as:

$$(bx+c)\exp(-x) = a, \tag{A2}$$

where $c > b > a$ and $x > 0$. If $a < 0$, and $b \geq 0$, obviously solution does not exist because $x$ is positive. If $a < 0$, and $b < 0$, let us write Equation (A2) as:

$$\left(\frac{b}{a}x + \frac{c}{a}\right)\exp(-x) = 1, \tag{A3}$$

where $0 < b/\,a < 1$. The maximum value of the left-hand side of Equation (A3) is:

$$\frac{b}{a}\exp\left(\frac{c}{b}-1\right) < 1$$

Thus it follows that $b/a < 1$, and $c/b < 1$ since $b < 0$.

### *Appendix 3*

For the exponential distribution, we have the iterative relation:

$$\Delta_{n+1} = -\eta + \tau_0(\exp(\Delta_n/\tau_0) - 1).$$

This gives the expected cycle time, as a function of $\Delta_1$, the first inspection time:

$$\sum_{n=0}^{\infty} \Delta_{n+1}\exp(-T_n/\tau_0) = \Delta_1 + \tau_0 - \eta N_{\text{fa}},$$

where $N_{\text{fa}} = \sum_{n=1}^{\infty}\exp(-T_n/\tau_0)$ is the expected number of false alarms. Using the definition of $\eta$ in Equation (34), we can derive,

$$\eta = \frac{(\Delta_1+\tau_0)C_{\text{fa}}}{\delta\mu\tau_0 C_{\text{ur}} - C_{\text{mr}}},$$

and consequently,

$$AC_{\text{inp}} = \delta\mu C_{\text{ur}} - \frac{\delta\mu\tau_0 C_{\text{ur}} - C_{\text{mr}}}{\Delta_1+\tau_0}.$$

This shows that to reduce the average cost, $\Delta_1$ should be reduced to its lower bound, $\Delta^*$.

## Biographies

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