Technical Appendix
Used Goods, Not Used Bads:
Profitable Secondary Market Sales for a Durable Goods Channel

Equilibrium Quantities for Integrated Channel with Commitments to Consumers. (See Table 1)

The channel's objective is given by:

$$\max_{q_{2n},q_{2u},q_{1n}} \pi^2_{channel} = (p_{2n}-c)q_{2n} + (p_{2u}-c_u)q_{2u} + (p_{1n}-c)q_{1n}$$
$$s.t.\ \{q_{2n},q_{2u},q_{1n}\} \ge 0,\ q_{1n} \ge q_{2u}$$

We will verify that $q_{2u} \ge 0$ and $q_{1n} \ge 0$ after developing the following Lagrangian:

$$L(q_{2n},q_{2u},q_{1n},\lambda,\psi;\alpha,\theta,c) = (p_{2u}-c_u)q_{2u} + (p_{2n}-c)q_{2n} + (p_{1n}-c)q_{1n} + \lambda q_{2n} + \psi(q_{1n}-q_{2u})$$

Substituting for the inverse demand equations, we derive the Kuhn-Tucker conditions:

$$\frac{\partial L(q_{2n},q_{2u},q_{1n},\lambda_1,\psi;\alpha,\theta,c,\gamma)}{q_{2n}} = -c - 2q_{2n}(1+\alpha) + \gamma + \alpha\gamma - 2q_{2u}(\alpha+\theta) + \lambda = 0$$

$$\frac{\partial L(q_{2n},q_{2u},q_{1n},\lambda_1,\psi;\alpha,\theta,c,\gamma)}{q_{2u}} = \alpha(-1+2q_{1n}-2q_{2n}-4q_{2u}+\gamma) + \theta(\gamma - 2(q_{2n}+q_{2u})) - \psi = 0$$

$$\frac{\partial L(q_{2n},q_{2u},q_{1n},\lambda_1,\psi;\alpha,\theta,c,\gamma)}{q_{1n}} = 1 - c + \alpha + 2\alpha q_{2u} - 2(1+\alpha)q_{1n} + \psi = 0$$

$$\lambda q_{2n} = 0$$
$$\psi(q_{1n}-q_{2u}) = 0$$
$$\lambda \ge 0$$
$$\psi \ge 0$$
$$q_{2n} \ge 0$$
$$q_{1n} \ge q_{2u}$$

Solving for $q_{2n}, q_{2u}, q_{1n}, \lambda$, and ψ yields four cases:

CASE 1: $q_{2n} > 0$, $q_{1n} > q_{2u}$

$$\lambda^* = 0$$
$$\psi^* = 0$$
$$q^*_{2n} = \frac{1}{2}\left(\gamma - \frac{c(2\alpha+\theta)}{(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}\right)$$
$$q^*_{2u} = \frac{c\theta}{2(2\alpha-\alpha\theta+\theta-\theta^2)}$$
$$q^*_{1n} = \frac{1+\alpha-c}{2(1+\alpha)} + \frac{c\alpha\theta}{2(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}$$

CASE 2: $q_{2n} = 0$, $q_{1n} > q_{2u}$

$$\lambda^* = \frac{c(2\alpha+\theta)+\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{\theta+\alpha(2+\alpha+\theta)}$$
$$\psi^* = 0$$
$$q_{2n}^* = 0$$
$$q_{2u}^* = \frac{\gamma(1+\alpha)(\alpha+\theta)-c\alpha}{2(\theta+2\alpha+\alpha^2+\alpha\theta)}$$
$$q_{1n}^* = \frac{\alpha(2-2c+\alpha+\alpha\gamma)+\theta(1-c+\alpha+\alpha\gamma)}{2(\theta+2\alpha+\alpha^2+\alpha\theta)}$$

CASE 3: $q_{2n} = 0$, $q_{1n} = q_{2u}$

$$\lambda^* = 1-2\gamma+\gamma\theta+\frac{c+\gamma-1}{1+\alpha+\theta}$$
$$\psi^* = \frac{c+\gamma-1}{1+\alpha+\theta}-(1+\alpha-c-\gamma)$$
$$q_{2n}^* = 0$$
$$q_{2u}^* = \frac{1-c+\gamma(\alpha+\theta)}{2(1+\alpha+\theta)}$$
$$q_{1n}^* = \frac{1-c+\gamma(\alpha+\theta)}{2(1+\alpha+\theta)}$$

CASE 4: $q_{2n} > 0$, $q_{1n} = q_{2u}$

$$\lambda^* = 0$$
$$\psi^* = \frac{1+\alpha-c+c\theta}{1+2\alpha-\alpha\theta+\theta-\theta^2}-1-\alpha+c$$
$$q_{2n}^* = \frac{1}{2}\left(\gamma-\frac{c+\alpha+\theta}{1+2\alpha-\alpha\theta+\theta-\theta^2}\right)$$
$$q_{2u}^* = \frac{1+\alpha-c+c\theta}{2(1+2\alpha-\alpha\theta+\theta-\theta^2)}$$
$$q_{1n}^* = \frac{1+\alpha-c+c\theta}{2(1+2\alpha-\alpha\theta+\theta-\theta^2)}$$

From the restriction that $\{\lambda,\psi\} > 0$, we determine when each of the cases is feasible. Under the assumption that $\frac{\alpha}{\alpha+\theta} < \gamma < \frac{2\alpha+\theta}{(2-\theta)(\alpha+\theta)}$, CASE 4 cannot exist because $\psi^* > 0$ implies that $c > \frac{(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{(2-\theta)(\alpha+\theta)}$. When q_{2n}^* is evaluated at its upper bound on $\gamma = \frac{2\alpha+\theta}{(2-\theta)(\alpha+\theta)}$ and this lower bound on *c*, it is equal to zero. The expression for q_{2n}^* is decreasing in γ and increasing in *c*. Therefore, the region for which $\psi^* > 0$ precludes q_{2n}^* from being positive and both non-negativity constraints cannot be satisfied over the parameter space.

Examining q_{2n}^* of CASE 1, we see CASE 1 is feasible if and only if

$$c < c^*(\gamma,\alpha,\theta) \equiv \frac{\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{2\alpha+\theta}.$$

Examining λ^* and (q_{1n}^* - q_{2u}^*) of CASE 2, we see CASE 2 is feasible if and only if

$$c^*(\gamma,\alpha,\theta) \equiv \frac{\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{2\alpha+\theta} < c < c^{**}(\gamma,\alpha,\theta) \equiv 1+\alpha-\gamma+\frac{\alpha}{\alpha+\theta}.$$

Examining λ^* of CASE 3, we see that CASE 3 is feasible if and only if

c> $c^{**}(\gamma,\alpha,\theta) \equiv 1+\alpha-\gamma+\dfrac{\alpha}{\alpha+\theta}$.

In summary:

If $c < c^*(\gamma,\alpha,\theta) \equiv \dfrac{\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{2\alpha+\theta}$, CASE 1 is the equilibrium.

If $c^*(\gamma,\alpha,\theta) \equiv \dfrac{\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{2\alpha+\theta} < c < c^{**}(\gamma,\alpha,\theta) \equiv 1+\alpha-\gamma+\dfrac{\alpha}{\alpha+\theta}$, CASE 2 is the equilibrium.

If c> $c^{**}(\gamma,\alpha,\theta) \equiv 1+\alpha-\gamma+\dfrac{\alpha}{\alpha+\theta}$, CASE 3 is the equilibrium.

Equilibrium prices follow directly from substituting the equilibrium quantities into equations (1)-(4) of the text. Equilibrium profits follow directly from substituting the equilibrium quantities into the firm's profit expression.

Q.E.D.

Proof that $\Gamma^*(q_{1n}) = \dfrac{(\gamma\theta-\alpha+\alpha q_{1n}+\alpha\gamma)^2}{4(2\alpha+\theta)}$ **(See Equation 6)**

$\Gamma = (p_{2u}-c_u)q_{2u}$=$[(\gamma-q_{2u})(\alpha+\theta)-(1-q_{1n}+q_{2u})\alpha]q_{2u}$ if $q_{2n}=0$.

First order condition with respect to q_{2u}: $\alpha(\gamma+q_{1n}-1-4q_{2u})-2\theta q_{2u}+\gamma\theta=0$

Second order condition with respect to q_{2u}: $-4\alpha-2\theta<0$

Thus $q_{2u} = \dfrac{\gamma\theta-\alpha+\alpha q_{1n}+\alpha\gamma}{2(2\alpha+\theta)}$ maximizes Γ.

Substituting $q_{2u} = \dfrac{\gamma\theta-\alpha+\alpha q_{1n}+\alpha\gamma}{2(2\alpha+\theta)}$ into $[(\gamma-q_{2u})(\alpha+\theta)-(1-q_{1n}+q_{2u})\alpha]q_{2u}$ yields

$$\Gamma^*(q_{2n}) = \frac{(\gamma\theta-\alpha+\alpha q_{1n}+\alpha\gamma)^2}{4(2\alpha+\theta)}$$

Q.E.D.

Proposition 1: *For* $c < c^*(\gamma,\alpha,\theta)$, *the following multi-part tariff coordinates the distribution channel:*

$$w_1^* = \frac{1}{4}\left[4c + 2\alpha\gamma - \frac{\alpha^2(1+2\gamma)}{2\alpha+\theta} - \frac{c\alpha(\alpha+\theta)}{(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)} + \frac{\alpha(\alpha - c\alpha + \alpha^2 - c\theta)}{\alpha^2+2\alpha+\theta-\alpha\theta-\theta^2}\right]$$

$$w_2^* = c + \frac{\alpha(1+\alpha)(1-c)}{2(\alpha+\theta)} + \frac{c\alpha^2(2-\theta)}{2(2\alpha+\theta-\alpha\theta-\theta^2)}$$

$$P_{2n}^* = \frac{\gamma+\alpha\gamma+c}{2}$$

$$F_1^* = (p_{1n}^* - w_1^*)q_{1n}^* + (p_{2n}^* - w_2^*)q_{2n}^* + (p_{2u}^* - c_u^*)q_{2u}^* - t * p_{2n} - F_2^* - \underline{\pi}^{ret}$$

$$F_2^* = (p_{2n}^* - w_2^*)q_{2n}^* + (p_{2u}^* - c_u^*)q_{2u}^* - \Gamma^*.$$

This tariff replicates the outcomes of a vertically-integrated firm that can make credible quantity commitments to consumers. In this contract, the manufacturer charges $w_1^* > c$ *and a fixed fee* F_1^* *in the first period. In the second period, the manufacturer charges* $w_2^* > c$ *and a fixed fee,* F_2^**, in addition to imposing a maximum resale price,* P_{2n}^*. *The retailer makes only a normal return to its capital, while the manufacturer garners the remainder of total channel profits.*

Proof:

We look at the case $0 < c < c^*(\alpha,\theta) \equiv \dfrac{\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{2\alpha+\theta}$.

With a price ceiling, the retailer's objective in the second period is to maximize profit, such that the retail price for new goods is less than the price ceiling, P_{2n}. Recall p_{tj} and q_{tj} are the retail prices and quantities of good type j in period t, F_t is the fixed fee payment from the retailer to the manufacturer in period t, and w_t is the per-unit wholesale price in period t. This yields the constrained optimization problem with Lagrangian multiplier, λ, below:

$$\max_{q_{2n},q_{2u}} (p_{2n} - w_2)q_{2n} + (p_{2u} - c_u)q_{2u} - F_2 + \lambda(P_{2n} - p_{2n})$$

$$\begin{aligned} s.t \quad & \lambda \geq 0 \\ & (P_{2n} - p_{2n}) \geq 0 \\ & \lambda(P_{2n} - p_{2n}) = 0 \end{aligned}$$

Solving the Kuhn-Tucker conditions yields two possible equilibria:

A1:

$$\lambda = \frac{\gamma + \alpha\gamma - 2P_{2n} + w_2}{1+\alpha}$$

$$q_{2n}(q_{1n}, P_{2n}, w_2; \gamma, \alpha, \theta) = \frac{w_2 + 2(1+\alpha)\gamma - 2P_{2n}}{2(1+\alpha)} - \frac{w_2(2\alpha+\theta) - \alpha(1-q_{1n})(\alpha+\theta)}{2(\alpha^2 + 2\alpha - \alpha\theta + \theta - \theta^2)}$$

$$q_{2u}(q_{1n}, P_{2n}, w_2; \gamma, \alpha, \theta) = \frac{w_2\theta + \alpha(w_2 - 1 + q_{1n} + \alpha q_{1n} - \alpha)}{2(\alpha^2 + 2\alpha - \alpha\theta + \theta - \theta^2)}$$

A2:

$$\lambda = 0$$

$$q_{2n}(q_{1n}, P_{2n}, w_2; \gamma, \alpha, \theta) = \frac{1}{2}\left(\gamma - \frac{w_2(2\alpha+\theta) - \alpha(1-q_{1n})(\alpha+\theta)}{(\alpha^2 + 2\alpha - \alpha\theta + \theta - \theta^2)}\right)$$

$$q_{2u}(q_{1n}, P_{2n}, w_2; \gamma, \alpha, \theta) = \frac{w_2\theta + \alpha(w_2 - 1 + q_{1n} + \alpha q_{1n} - \alpha)}{2(\alpha^2 + 2\alpha - \alpha\theta + \theta - \theta^2)}$$

with A1 being the solution if and only if P_{2n} and w_2 are chosen by the manufacturer such that $1 - 2P_{2n} + w_2 > 0$. As shown in the text, F_2 will be chosen such that the retailer's second period profit is reduced to $\Gamma^* = \frac{(\gamma\theta - \alpha + \alpha q_{1n} + \alpha\gamma)^2}{4(2\alpha+\theta)}$. Therefore, in the first period, the retailer's optimization problem is to

$$\max_{q_{1n}} \pi_{1and\,2}^{ret} = (p_{1n} - w_1)q_{1n} - F_1 + \Gamma^*(q_{1n})$$
$$s.t.\ E(q_{2u}) = q_{2u}(q_{1n}, P_{2n}, w_2; \gamma, \alpha, \theta)$$

From the first order condition, the retailer's first period reaction to the manufacturer's contract is given by

$$q_{1n}(P_{2n}, w_1, w_2; \gamma, \alpha, \theta) = \frac{\alpha^4(1+\gamma) + 2\alpha^3(4 - 2w_1 + w_2 + \gamma - \theta) + 2\theta^2(1-w_1)(1-\theta))}{3\alpha^4 + \alpha^3(18 - 5\theta) + 4\theta^2(1-\theta) + \alpha^2(16 + 9\theta - 11\theta^2 + 4\alpha\theta(4 - 2\theta - \theta^2))} +$$

$$\frac{\alpha^2(8 - 2w_1(4-\theta) + \theta(4 + 3w_2 + 3\gamma - 5\theta - 2\gamma\theta)) + \alpha\theta(8 - 8w_1 + 6\theta w_1 - \theta(4 - w_2 - \gamma + \theta(2+\gamma)))}{3\alpha^4 + \alpha^3(18 - 5\theta) + 4\theta^2(1-\theta) + \alpha^2(16 + 9\theta - 11\theta^2 + 4\alpha\theta(4 - 2\theta - \theta^2))}$$

for either of the two equilibrium retailer choices in the second period.

As shown in the text, F_1 will be chosen such that

$F_1 = (p_{1n} - w_1)q_{1n} + (p_{2n} - w_2)q_{2n} + (p_{2u} - c_u)q_{2u} - F_2 - \underline{\pi}^{ret}$.

The manufacturer's objective is:

$\max_{P_{2n},w_1,w_2} (p_{2u} - c_u)q_{2u} + (p_{2n} - c)q_{2n} + (p_{1n} - c)q_{1n} - \underline{\pi}^{ret}$,

where p_{tj} and c_u are given by Equations (1)-(4) and q_{tj} are $q_{tj}(P_{2n}, w_1, w_2; \alpha, \theta, c)$ from above. We first look at the equilibrium in the second period with

$\lambda = \frac{\gamma + \alpha\gamma - 2P_{2n} + w_2}{1+\alpha}$.

Solving the first-order conditions yields:

$$w_1^* = \frac{1}{4}\left[4c + 2\alpha\gamma - \frac{\alpha^2(1+2\gamma)}{2\alpha+\theta} - \frac{c\alpha(\alpha+\theta)}{(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)} + \frac{\alpha(\alpha - c\alpha + \alpha^2 - c\theta)}{\alpha^2 + 2\alpha + \theta - \alpha\theta - \theta^2}\right]$$

$$w_2^* = c + \frac{\alpha(1+\alpha)(1-c)}{2(\alpha+\theta)} + \frac{c\alpha^2(2-\theta)}{2(2\alpha+\theta-\alpha\theta-\theta^2)}$$

$$P_{2n}^* = \frac{\gamma + \alpha\gamma + c}{2}.$$

The above contract is a candidate equilibrium. Substituting this contract into the retailer's reaction yields $\lambda^* = \frac{\alpha}{\alpha+\theta} > 0$ for all c and $\alpha, \theta > 0$. The equilibrium response by the retailer to this contract is then bound by P_{2n}^* and is given below:

$$q_{1n}^* = \left(\frac{\theta(1 - c - \theta + c\theta) + \alpha(2 - 2c + 2\alpha + 2c\theta - \theta^2 - \alpha\theta)}{2(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)}\right)$$

$$q_{2u}^* = \frac{c\theta}{2(2\alpha+\theta-\alpha\theta-\theta^2)}$$

$$q_{2n}^* = \frac{1}{2}\left(\gamma - \frac{c(2\alpha+\theta)}{(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)}\right).$$

This response has the same quantities as chosen by the vertically integrated channel with quantity commitment as presented in Table 1 of the text. The manufacturer will then choose to offer this contract over any other because the manufacturer's objective, $(p_{2u} - c_u)q_{2u} + (p_{2n} - c)q_{2n} + (p_{1n} - c)q_{1n} - \underline{\pi}^{ret}$, is maximized at the quantities chosen by the vertically integrated channel with quantity commitment achieved by the above contract. Thus,

$$q_{1n}^* = (\frac{\theta(1-c-\theta+c\theta)+\alpha(2-2c+2\alpha+2c\theta-\theta^2-\alpha\theta)}{2(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)})$$
$$q_{2u}^* = \frac{c\theta}{2(2\alpha+\theta-\alpha\theta-\theta^2)}$$
$$q_{2n}^* = \frac{1}{2}(\gamma - \frac{c(2\alpha+\theta)}{(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)})$$

is the unique equilibrium when the manufacturer offers a price ceiling.
We see the constraint that the constraint $q_{2n} = \frac{1}{2}(\gamma - \frac{c(2\alpha+\theta)}{(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)})>0$ is satisfied if and only if $c < c^*(\alpha,\theta) \equiv \frac{\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{2\alpha+\theta}$. Also,

$q_{1n} - q_{2u} = \frac{1}{2} - \frac{c(2-\theta)(\alpha+\theta)}{(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}>0$ for all $c<\frac{(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)}{(2-\theta)(\alpha+\theta)}$.

Because $\frac{(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)}{(2-\theta)(\alpha+\theta)} > c^*(\alpha,\theta)$ for all $\frac{\alpha}{\alpha+\theta} < \gamma < \frac{2\alpha+\theta}{(2-\theta)(\alpha+\theta)}$, it is also true that $q_{1n} > q_{2u}$ for all $c < c^*(\alpha,\theta)$.

The fixed fees are presented in their entirety below:

$$F_1^* =$$

$$4\alpha^4(c^2(16+3\alpha(6+\alpha))-2c(1+\alpha)(16+3\alpha(6+\alpha))+(1+\alpha)^2(16+\alpha(26+8\gamma(-2+\gamma)+$$
$$\alpha(7+4\gamma(-2+\gamma)))))-4\alpha^3(c^2(-32+\alpha(5+38\alpha+6\alpha^2))-c(1+\alpha)(-64+\alpha(-6+\alpha(58+9\alpha)))$$
$$+(1+\alpha)^2(-32+\alpha(-23+7\alpha(4+\alpha)+40\gamma-8\alpha\gamma(1+\alpha)+4\gamma^2(-7+\alpha^2))))\theta+$$
$$\alpha^2(-2c(1+\alpha)(96+\alpha(-186+\alpha(209+45\alpha+6\alpha^2)))+c^2(96+\alpha(-234+\alpha(-213+4\alpha(22+3\alpha))))$$
$$+(1+\alpha)^2(96+\alpha(-114-281\alpha+20\alpha^2+7\alpha^3-8(18+\alpha(-31+\alpha(-4+\alpha)))\gamma$$
$$+4(38+\alpha(-35+\alpha(-8+\alpha)))\gamma^2)))\theta^2+\alpha(2c(1+\alpha)(-32+\alpha(207+\alpha(36+\alpha(-167+\alpha))))$$
$$+c^2(32+\alpha(-231+\alpha(19+230\alpha-8\alpha^2)))+(1+\alpha)^2(32+\alpha(-179+5\alpha(-27+\alpha(33+\alpha))-56\gamma$$
$$-8\alpha(-38+\alpha(13+2\alpha))\gamma+4(25+3\alpha(-23+\alpha(3+\alpha)))\gamma^2)))\theta^3$$
$$+(c^2(4+\alpha(-88+\alpha(147+\alpha(152-61\alpha))))+2c(1+\alpha)(-4+\alpha(84+\alpha(-107+2\alpha(-70+17\alpha))))$$
$$+(1+\alpha)^2(4+\alpha(\alpha(\alpha(59+2(84-13\alpha)\alpha)+8\alpha(18+\alpha(-25+\alpha))\gamma+4(8+\alpha(-56+\alpha(38+\alpha)))\gamma^2$$
$$-8)10+\gamma))))\theta^4-(c^2(12+\alpha(-76+5\alpha(-3+13\alpha)))-2c(1+\alpha)(12+\alpha(-68+\alpha(-27+46\alpha)))$$
$$+(1+\alpha)^2(12-4\gamma^2+12\alpha(-1+\gamma)(5+7\gamma)+\alpha^2(-47+40(3-4\gamma)\gamma+\alpha^3(43+8\gamma(-5+3\gamma))))\theta^5$$
$$-(c(24-2\alpha^2(35+23\alpha))+c^2(-12+\alpha(16+27\alpha))+(1+\alpha)^2(-12+\alpha(8+23\alpha)-8\alpha\gamma(-3+4\alpha)$$
$$+12(1+3(-2+\alpha)\alpha)\gamma^2))\theta^6-4(1+\alpha)((1-c+\alpha)^2-2\alpha\gamma(1+\alpha)+(1+\alpha)(-3+5\alpha)\gamma^2)\theta^7-4\gamma^2\theta^8(1+\alpha)^2)$$
$$\div(16(1+\alpha)^2(2\alpha+\theta)(\alpha(-2+\theta)-\theta+\theta^2)(\alpha(2+\alpha)+\theta-\alpha\theta-\theta^2))$$

$$F_2^* =$$

$$\frac{-1}{16(\alpha+\theta)}(\alpha^2(1+4\gamma)-4\gamma^2(\theta-\theta^2)-\frac{\alpha^3(1+2\gamma)^2}{2\alpha+\theta}+4\alpha\gamma(1-\gamma-\theta)+$$
$$\frac{c^2(2\alpha+\theta)(\alpha^3-\alpha\theta(8+\alpha(11+6\alpha))+\theta^2(-4+\alpha^2(2+\alpha))+\theta^3(2+\alpha)^2)}{(1+\alpha)^2(2\alpha-\alpha\theta+\theta-\theta^2)^2}$$
$$+\frac{2c(-\alpha(2+\alpha-2\gamma)+2\gamma\theta(2+\alpha)+\frac{\alpha^2(4\gamma\theta-\alpha(1-2\gamma(1+\theta))}{2\alpha-\alpha\theta+\theta-\theta^2}}{1+\alpha})$$

Q.E.D.

The equilibrium retail prices are presented below:

$$p_{1n}=\frac{1+c+\alpha}{2}$$
$$p_{2n}=\frac{\gamma+c+\alpha\gamma}{2}$$
$$p_{2u}=\frac{\theta(\theta-\theta^2+\alpha(1+2c-c\theta+\theta-\theta^2))}{2(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)}$$
$$c_u=\frac{\alpha^2(2+2c+2\alpha-c\theta-\alpha\theta-\theta^2)+\alpha(\theta+2c\theta+\theta^2+c\theta^2)}{2(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)}.$$

Proof that F2 may be negative.

Evaluating F_2^* at the upper bound on $c = c^*(\gamma,\alpha,\theta) \equiv \frac{\gamma(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}{2\alpha+\theta}$, we have

$$F_2^* = -\frac{\alpha^2(1-\gamma+\gamma\theta)^2}{16(2\alpha+\theta)} < 0.$$

Q.E.D.

Proof that $w_1 - c$ is decreasing in *c*.

$$\frac{\partial(w_1-c)}{\partial c} = \frac{\alpha^2+\alpha\theta}{4}\left(\frac{-1}{(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)} - \frac{1}{\alpha^2+2\alpha-\alpha\theta+\theta-\theta^2}\right) < 0$$ by the fact that $\alpha < \theta < 1$.

Q.E.D.

Proof that $w_1 > c$.

$w_1 - c$ is decreasing in *c*. Evaluated at $c^*(\gamma,\alpha,\theta)$ this expression is equal to $\frac{\alpha^2(1-\gamma+\gamma\theta)(\alpha+\theta)^2}{4(2\alpha+\theta)(\alpha^2+2\alpha-\alpha\theta+\theta-\theta^2)} > 0$. Therefore, for all $c < c^*(\gamma,\alpha,\theta)$, the value $w_1 - c$ is positive, implying $w_1 > c$.

Q.E.D.

Proof that potential gains from renegotiating and ignoring first-period consumer expectations can be eliminated by the loss in profit due to re-formed second-period consumer expectations. (See Section 2.2)

The retailer and the manufacturer will choose to renegotiate if renegotiation can make both players better off. Therefore, renegotiation will occur only if total channel profit can be greater. Let s_h be the value of the future relationship if renegotiation does not occur. Let s_l denote the value of the future relationship if renegotiation does occur. Let $s_h - s_l > 0$, reflecting the fact that consumers rationally update to lower expectations of future buy-back price if there is renegotiation, with a resulting negative impact on channel profits. If $c < c^*(\gamma,\alpha,\theta)$, the maximum total channel profit possible in the second period, if the original choices made under commitment are discarded, is $\max_{q_{2n},q_{2u}} q_{2n}(p_{2n}-c) + q_{2u}(p_{2u}-c_u)$ where p_{2n} and p_{2u} are given by Equations 3 and 4 in the text and the value of $q_{1n}^* = \left(\frac{\theta(1-c-\theta+c\theta)+\alpha(2-2c+2\alpha+2c\theta-\theta^2-\alpha\theta)}{2(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)}\right)$, as generated by the equilibrium contract from the text. Therefore, the maximum channel profit when renegotiation occurs is:

$$\pi_2^{renegotiate} = \frac{1}{16(1+\alpha)(-2\alpha+\alpha\theta-\theta+\theta^2)^2(\alpha^2+2\alpha+\theta-\alpha\theta-\theta^2)}[-2c(1+\alpha)(\alpha(-2+\theta)+(-1+\theta)\theta)*$$

$$(\alpha^3(-2+4\gamma(-2+\theta))-2\alpha\theta(4\gamma(-2+\theta)-\theta)(-1+\theta)-4\gamma\theta^2(1-\theta)^2+\alpha^2(\theta(3\theta-5)+4\gamma(-4+3\theta)))+$$

$$+(1+\alpha)^2(-2\alpha+\alpha\theta-\theta+\theta^2)^2(\alpha(\alpha+4(2+\alpha)\gamma^2)-4(\alpha-1)\gamma^2\theta-4\gamma^2\theta^2)$$

$$+c^2(2\alpha+\theta)(8\alpha\theta(2-\theta)(1-\theta)+4\theta^2(1-\theta)^2+2\alpha^3(5-4\theta+\theta^2)+\alpha^2(16-11\theta-2\theta^2+\theta^3))].$$

The difference between renegotiation-enabled channel profit in the second period and the second-period channel profit with manufacturer commitment is

$$\frac{\alpha^2(\alpha^2(-2+\theta)+\theta(-1+c+\theta-c\theta)+\alpha(-2-2c(-1+\theta)+\theta^2))^2}{16(1+\alpha)(-2\alpha+\alpha\theta-\theta+\theta^2)^2(\alpha^2+2\alpha+\theta-\alpha\theta-\theta^2)}.$$

Because there is no logical upper limit on s_h - s_l, there exists an

$$(s_h-s_l)^* \equiv \frac{\alpha^2(\alpha^2(-2+\theta)+\theta(-1+c+\theta-c\theta)+\alpha(-2-2c(-1+\theta)+\theta^2))^2}{16(1+\alpha)(-2\alpha+\alpha\theta-\theta+\theta^2)^2(\alpha^2+2\alpha+\theta-\alpha\theta-\theta^2)}$$

such that for all $(s_h-s_l)\geq(s_h-s_l)^*$ the contract will not be renegotiated after the first period.
If $c\geq c^*(\gamma,\alpha,\theta)$, there is no profitable renegotiation in the second period.

Q.E.D.

Proposition 2: *For c> $c^*(\gamma,\alpha,\theta)$, in equilibrium, the manufacturer charges w_1^* equal to the marginal cost of production, c, and a fixed fee F_1 in the first period, and ceases production in the second period, in which the retailer sells only used goods. If no first-period consumers keep their purchase in the second period, that is, $c\geq\bar{c}(\gamma,\alpha,\theta)\geq c^*(\gamma,\alpha,\theta)$, then the maximum profits of an integrated channel are achieved. If any first-period consumers keep their purchase in the second period, the equilibrium outcome nets the channel lower profits than if credible commitments to consumers were feasible.*

Proof:

We first derive the equilibrium contract and resulting quantities. We then compare these quantities and the channel profit to that of the vertically-integrated channel with quantity commitments.

As shown in the proof of Proposition 2, the manufacturer will cease new good production in the second period and yield to the used good market if $c > c^*(\alpha,\theta)$. The sequence of events becomes:

1) The manufacturer offers contract of fixed fee, F_1, and per-unit wholesale price, w_1.
2) The retailer chooses first period quantity.
3) Consumers make purchase decision according to equation (2) in the text.
4) The retailer chooses second period used good quantity to sell at the price governed by equation (3) and purchased at a cost governed by equation (1) with $q_{2n} = 0$.

In the second period, the retailer sells used goods, subject to availability. This yields the retailer's constrained optimization problem with Lagrangian multiplier, λ, below:

$$\begin{aligned}&\max_{q_{2u}}(p_{2u} - c_u)q_{2u} + \lambda(q_{1n} - q_{2u})\\ &s.t \quad \lambda \geq 0\\ &\qquad (q_{1n} - q_{2n}) \geq 0\\ &\qquad \lambda(q_{1n} - q_{2n}) = 0\end{aligned}$$

Solving the Kuhn-Tucker conditions, we have the following possible outcomes:

B1:

$\lambda = 0$

$$q_{2u}(q_{1n};\gamma,\alpha,\theta) = \frac{\alpha(-1 + q_{1n} + \gamma) + \gamma\theta}{2(2\alpha + \theta)}$$

B2:

$\lambda(q_{1n};\gamma,\alpha,\theta) = \alpha(\gamma - 1 - 3q_{1n}) + \theta(\gamma - 2q_{1n})$

$q_{2u}(q_{1n}) = q_{1n}$,

with B1 occurring if $q_{1n} \geq \dfrac{-\alpha + \gamma(\alpha + \theta)}{3\alpha + 2\theta}$ and B2 occurring otherwise.

We now look at the firm's sub-game perfect equilibrium choice of q_{1n} in the first period.

In a rational expectations equilibrium, the retailer's choice of q_{1n} will maximize first and second period combined profits:

$$\max_{q_{1n}} (p_{1n} - w_1)q_{1n} + (p_{2u} - c_u)q_{2u} - F_1$$
$$s.t. \quad q_{2u}(q_{1n};\gamma,\alpha,\theta) = E(q_{2u}) \equiv I*(\arg\max_{q_{2u}} \pi_2^{ret}) + (1-I)q_{1n}$$

where I is an indicator variable taking the value 1 if $q_{1n} \geq \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}$ and 0 otherwise. We look at the two cases of the indicator function separately and then identify when each will happen.

For B1, some first-period consumers decide to keep their purchase into the second period. This implies that the indicator function I takes the value of 1 and the retailer's objective is:

$$\max_{q_{1n}} (p_{1n} - w_1)q_{1n} + q_{2u}(q_{1n};\gamma,\alpha,\theta)*(p_u(q_{1n};\gamma,\alpha,\theta) - c_u(q_{1n};\gamma,\alpha,\theta)) + \lambda_2(q_{1n} - \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}) - F_1$$
$$s.t. \quad \lambda_2 \geq 0$$
$$q_{1n} \geq \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}$$
$$\lambda_2(q_{1n} - \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}) = 0$$

where $q_{2u}(q_{1n};\gamma,\alpha,\theta)$ is defined from B1 above and $p_u(q_{1n};\gamma,\alpha,\theta)$ and $c_u(q_{1n};\gamma,\alpha,\theta)$ are derived from plugging $q_{2u}(q_{1n};\gamma,\alpha,\theta)$ into equations (2) and (3).
Solving the Kuhn-Tucker conditions, we again have two possible cases as below:

B1i:

$$\lambda_2 = 0$$
$$q_{1n} = \frac{2\alpha(2-2w_1+\alpha+\alpha\gamma)+2\theta(1-w_1+\alpha+\alpha\gamma)}{5\alpha^2+8\alpha+4\theta+4\alpha\theta}$$

B1ii:

$$\lambda_2 = \frac{1}{2}(\frac{\alpha(1+\gamma)}{2\alpha+\theta} - \frac{\alpha(2-\alpha)(2+\gamma)}{(3\alpha+2\theta)^2} + \frac{2(-1+w_1+\gamma-2\alpha-\alpha\gamma)}{3\alpha+2\theta})$$
$$q_{1n} = \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}.$$

We examine the manufacturer's optimal contract. The retailer will accept the manufacturer's contract if it yields the retailer a profit equal to the value of the outside option $\underline{\pi}^{ret}$. Therefore, the manufacturer chooses w_1 and F_1 to maximize profits, subject to retailer's acceptance of the contract:

$$\max_{w_1,F_1}(w_1-c)q_{1n}(w_1;\gamma,\alpha\theta)+F_1$$
$$s.t.\quad F_1 \le (p_{1n}(w_1;\gamma,\alpha\theta)-w_1)*q_{1n}(w_1;\gamma,\alpha\theta)+(p_{2u}(w_1;\gamma,\alpha\theta)-c_u(w_1;\gamma,\alpha\theta))*q_{2u}(w_1;\gamma,\alpha\theta)-\overline{\pi}^{ret}$$

where $q_{2u}(w_1;\gamma,\alpha\theta)$ and $q_{1n}(w_1;\gamma,\alpha\theta)$ are the retailer's reaction functions described by B1 and either B1i or B1ii. Because the manufacturer's profits are strictly increasing in F_1, this fixed fee will be charged such that the constraint binds. The manufacturer's objective function is then:

$$\max_{w_1}\pi^{mfgr}=(p_{1n}(w_1;\gamma,\alpha\theta)-c)*q_{1n}(w_1;\gamma,\alpha\theta)+(p_{2u}(w_1;\gamma,\alpha\theta)-c_u(w_1;\gamma,\alpha\theta))*q_{2u}(w_1;\gamma,\alpha\theta)-\overline{\pi}^{ret}.$$

For B1i:

The first order condition is

$$\frac{\partial\pi^{mfgr}}{\partial w_1}=\frac{2(c-w_1)(2\alpha+\theta)}{5\alpha^2+8\alpha+4\theta+4\alpha\theta}=0,$$

which is satisfied at

$$w_1^*=c.$$

Profit is extracted through the fixed fee equal to

$$F_1^*=\frac{(\alpha(-1+\gamma)+\gamma\theta)^2}{4(2\alpha+\theta)}+\frac{4(\alpha(2-2c+\alpha+\alpha\gamma)+\theta(1-c+\alpha+\alpha\gamma))^2}{4(2\alpha+\theta)(4\theta(1+\alpha)+8\alpha+5\alpha^2)}-\underline{\pi}^{ret}.$$

For B1ii, the manufacturer's profit objective function is $\frac{(\gamma\alpha-\alpha+\theta\gamma)(\alpha(4-3c+\alpha-\gamma+2\alpha\gamma)+\theta(2-2c+\alpha-\gamma+3\alpha\gamma)+\gamma\theta^2)}{(3\alpha+2\theta)^2}-\underline{\pi}^{ret}$, which is unaffected by choice of w_1. In which case, manufacturer profits will be maximized at any $\{w_1,F_1\}$ such that

$$F_1=(p_{1n}(w_1;\gamma,\alpha\theta)-w_1)*q_{1n}(w_1;\gamma,\alpha\theta)+(p_{2u}(w_1;\gamma,\alpha\theta)-c_u(w_1;\gamma,\alpha\theta))*q_{2u}(w_1;\gamma,\alpha\theta)-\underline{\pi}^{ret}$$
and $\frac{1}{2}\left(\frac{\alpha(1+\gamma)}{2\alpha+\theta}-\frac{\alpha(2-\alpha)(2+\gamma)}{(3\alpha+2\theta)^2}+\frac{2(-1+w_1+\gamma-2\alpha-\alpha\gamma)}{3\alpha+2\theta}\right)\ge 0$.

It must now be verified that the manufacturer would not prefer to increase the wholesale price above marginal cost in order to induce the retailer to choose B1ii.

The difference in profits from B1i and B1ii is equal to:

$$\frac{(\alpha^3(11+\gamma)-4\theta^2(\gamma-1+c)+2\alpha\theta(9-7c-6\gamma+2\theta)+\alpha^2(20-12c-8\gamma+\gamma\theta+14\theta))^2}{4(2\alpha+\theta)(3\alpha+2\theta)^2(5\alpha^2+8\alpha+4\theta+4\alpha\theta)}>0.$$

Therefore, the manufacturer, in equilibrium, will charge $w_1 = c$ for B1. The retailer will choose B1i if $\lambda_2 < 0$, which occurs if:

$c < 1+\alpha-\gamma+\frac{\alpha^2(1+\gamma)}{2(2\alpha+\theta)}+\frac{\alpha(2-\alpha)(2+\gamma)}{6\alpha+4\theta}$. The resulting quantities are in Table 2. If $c \geq 1+\alpha-\gamma+\frac{\alpha^2(1+\gamma)}{2(2\alpha+\theta)}+\frac{\alpha(2-\alpha)(2+\gamma)}{6\alpha+4\theta}$, the retailer prefers B1ii to B1i.

We now look at B2.

For B2, all first-period new goods are sold as used in the second period. This implies that the indicator function I takes the value of 0 and the retailer's objective is:

$$\begin{aligned}
&\max_{q_{1n}}(p_{1n}-w_1)q_{1n}+q_{1n}((\alpha+\theta)(\gamma-q_{1n})-\alpha)+\lambda_2(\frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}-q_{1n})-F_1\\
&s.t.\quad \lambda_2 \geq 0\\
&\qquad q_{1n} \leq \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}\\
&\qquad \lambda_2(\frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}-q_{1n})=0.
\end{aligned}$$

Solving the Kuhn-Tucker conditions, we have the following two possibilities:

B2i:

$$\lambda_2 = 0$$

$$q_{1n} = \frac{1-w_1+\gamma(\alpha+\theta)}{2(1+\alpha+\theta)}$$

B2ii:

$$\lambda_2 = \frac{\alpha^2(2+\gamma)-2\theta(\gamma+w_1-1)+\alpha(5-3w_1-2\gamma+2\theta+\gamma\theta)}{(3\alpha+2\theta)^2}$$

$$q_{1n} = \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}.$$

We examine the manufacturer's optimal contract. The retailer will accept the manufacturer's contract if it yields the retailer a profit equal to the value of the outside option $\bar{\pi}^{ret}$. Therefore, the manufacturer chooses w_1 and F_1 to maximize profits, subject to retailer's acceptance of the contract.

$$\max_{w_1, F_1}(w_1 - c)q_{1n}(w_1;\gamma,\alpha,\theta) + F_1$$

$$s.t.\quad F_1 \leq (p_{1n}(w_1;\gamma,\alpha,\theta) - w_1) * q_{1n}(w_1;\gamma,\alpha,\theta) + (p_{2u}(w_1;\gamma,\alpha,\theta) - c_u(w_1;\gamma,\alpha,\theta)) * q_{2u}(w_1;\gamma,\alpha,\theta) - \underline{\pi}^{ret}$$

where $q_{2u}(w_1;\gamma,\alpha,\theta)$ and $q_{1n}(w_1;\gamma,\alpha,\theta)$ are the retailer's reaction functions described by B2 and either B2i or B2ii. Because the manufacturer's profits are strictly increasing in F_1, this fixed fee will be charged such that the constraint binds. The manufacturer's objective function is then:

$$\max_{w_1} \pi^{mfgr} = (p_{1n}(w_1;\gamma,\alpha,\theta) - c) * q_{1n}(w_1;\gamma,\alpha,\theta) + (p_{2u}(w_1;\gamma,\alpha,\theta) - c_u(w_1;\gamma,\alpha,\theta)) * q_{2u}(w_1;\gamma,\alpha,\theta) - \overline{\pi}^{ret},$$

where the quantities, retail prices and buyback prices are derived from the retailer's reaction function from B2i and B2ii. The first order condition

$$\frac{\partial \pi^{mfgr}}{\partial w_1} = \frac{2(c - w_1)(2\alpha + \theta)}{5\alpha^2 + 8\alpha + 4\theta + 4\alpha\theta} = 0$$

is satisfied at

$$w_1^* = c.$$

The manufacturer extracts profits from its fixed fee,

$$F_1^* = \frac{(1 - c + \alpha\gamma + \theta\gamma)^2}{4(1 + \alpha + \theta)} - \overline{\pi}^{ret}.$$

Notice B2ii is the same as B1ii. It must now be verified that the manufacturer would not prefer to increase the wholesale price above marginal cost in order to induce the retailer to choose outcome B2ii. The manufacturer's profits from B2i minus the profits from B2ii is equal to

$$\frac{(2\alpha^2 + \alpha\gamma - 2\theta(\gamma - 1 + c) + \alpha(5 - 3c - 2\gamma + 2\theta + \gamma\theta))^2}{4(1 + \alpha + \theta)(3\alpha + 2\theta)^2} > 0.$$

Therefore, the manufacturer, in equilibrium, will charge $w_1 = c$ for B2. The retailer will choose B2i if $\lambda_2 < 0$, which occurs if $c > \frac{\alpha(5 - 2\gamma + 2\alpha + \alpha\gamma) + \theta(2 - 2\gamma + 2\alpha + \alpha\gamma)}{3\alpha + 2\theta}$. The resulting quantities are in Table 2. If $c \leq \frac{\alpha(5 - 2\gamma + 2\alpha + \alpha\gamma) + \theta(2 - 2\gamma + 2\alpha + \alpha\gamma)}{3\alpha + 2\theta}$, the retailer prefers B2ii to B2i.

We may now summarize the equilibrium:

If $c^*(\gamma,\alpha,\theta) \le c < 1+\alpha-\gamma+\frac{\alpha^2(1+\gamma)}{2(2\alpha+\theta)}+\frac{\alpha(2-\alpha)(2+\gamma)}{6\alpha+4\theta}$:

$$w_1^* = c$$

$$F_1^* = \frac{(\alpha(-1+\gamma)+\gamma\theta)^2}{4(2\alpha+\theta)}+\frac{(\alpha(2-2c+\alpha+\alpha\gamma)+\theta(1-c+\alpha+\alpha\gamma))^2}{(2\alpha+\theta)(4\theta(1+\alpha)+8\alpha+5\alpha^2)}-\underline{\pi}^{ret}$$

$$q_{1n}^* = \frac{2\alpha(2-2c+\alpha+\alpha\gamma)+2\theta(1-c+\alpha+\alpha\gamma)}{5\alpha^2+8\alpha+4\theta+4\alpha\theta}$$

$$q_{2u}^* = \frac{1}{6}(3\gamma-\frac{\alpha(1+\gamma)}{2\alpha+\theta}-\frac{2\alpha(1+3c+\alpha+\gamma(4+\alpha)}{\alpha(8+5\alpha)+4(1+\alpha)\theta})$$

If $1+\alpha-\gamma+\frac{\alpha^2(1+\gamma)}{2(2\alpha+\theta)}+\frac{\alpha(2-\alpha)(2+\gamma)}{6\alpha+4\theta} \le c \le \frac{\alpha(5-2\gamma+2\alpha+\alpha\gamma)+\theta(2-2\gamma+2\alpha+\alpha\gamma)}{3\alpha+2\theta}$:

$$w_1^* = c$$

$$F_1^* = \frac{(\gamma\alpha-\alpha+\theta\gamma)(\alpha(4-3c+\alpha-\gamma+2\alpha\gamma)+\theta(2-2c+\alpha-\gamma+3\alpha\gamma)+\gamma\theta^2)}{(3\alpha+2\theta)^2}-\underline{\pi}^{ret}$$

or any $\{w_1, F_1\}$ such that

$$F_1 = (p_{1n}(w_1;\gamma,\alpha\theta)-w_1)*q_{1n}(w_1;\gamma,\alpha\theta)+(p_{2u}(w_1;\gamma,\alpha\theta)-c_u(w_1;\gamma,\alpha\theta))*q_{2u}(w_1;\gamma,\alpha\theta)-\underline{\pi}^{ret}$$

and $\frac{1}{2}(\frac{\alpha(1+\gamma)}{2\alpha+\theta}-\frac{\alpha(2-\alpha)(2+\gamma)}{(3\alpha+2\theta)^2}+\frac{2(-1+w_1+\gamma-2\alpha-\alpha\gamma)}{3\alpha+2\theta}) \ge 0$.

$$q_{1n}^* = \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}$$

$$q_{2u}^* = \frac{-\alpha+\gamma(\alpha+\theta)}{3\alpha+2\theta}$$

If $c > \frac{\alpha(5-2\gamma+2\alpha+\alpha\gamma)+\theta(2-2\gamma+2\alpha+\alpha\gamma)}{3\alpha+2\theta}$:

$$w_1^* = c$$

$$F_1^* = \frac{(1-c+\alpha\gamma+\theta\gamma)^2}{4(1+\alpha+\theta)}-\pi^{ret}$$

$$q_{1n}^* = \frac{1-c+\gamma(\alpha+\theta)}{1+\alpha+\theta}$$

$$q_{2u}^* = \frac{1-c+\gamma(\alpha+\theta)}{2(1+\alpha+\theta)}$$

□

We now show that when there are first period consumers who keep their good into the second period, the decentralized channel earns lower profits than does the vertically integrated firm with quantity commitments. This is when $c^*(\gamma,\alpha,\theta) < c < \overline{c}(\gamma,\alpha,\theta)$.

The difference in profits of the vertically integrated firm with quantity commitments and the decentralized channel is equal to:
$\frac{\alpha^2(\alpha(2-2c+\alpha+\alpha\gamma)+\theta(1-c+\alpha+\alpha\gamma)^2}{4(2\alpha+\theta)(\alpha(8+5\alpha)+4\theta(1+\alpha))(\theta+\alpha(2+\alpha+\theta))} > 0$. Thus, for $c > c^*(\gamma,\alpha,\theta)$, the channel nets lower profits as a decentralized channel without commitments to consumers when there are first period consumers who keep their purchase into the second period.

Q.E.D.

Proof of Equilibrium Quantities for Decentralized Channel without a Secondary Market (See Table 3)

The inverse demand equations for new goods in each period are derived in a similar manner as in the first section. Without a used market, these prices are governed by the following functions:

$$p_{1n} = (1-q_{1n})(1+\alpha)$$
$$p_{2n} = (1-q_{2n})(1+\alpha).$$

The total channel profits are given by:

$$\pi_{channel} = (p_{1n}-c)q_{1n} + (p_{2n}-c)q_{2n}$$
$$s.t. \quad q_{2n} \geq 0.$$

This leads to the maximization problem with Lagrangian multiplier, χ,

$$\begin{aligned}\max_{q_{1n},q_{2n}} \; & L(q_{1n},q_{2n},\chi;\alpha,\theta,c) = (p_{1n}-c)q_{1n} + (p_{2n}-c)q_{2n} + \chi q_{2n} \\ s.t. \quad & \chi \geq 0 \\ & \chi q_{2n} = 0.\end{aligned}$$

We have the Kuhn-Tucker conditions below:

$$\begin{aligned}&1-c+\alpha-2q_{1n}(1+\alpha)=0\\&\chi+\alpha\gamma+\gamma-2q_{2n}(1+\alpha)-c=0\\&\chi q_{2n}=0\\&\chi\geq 0\end{aligned}$$

which are satisfied at:

$$\left.\begin{array}{l} q_{1n} = \dfrac{1-c+\alpha}{2(1+\alpha)} \\ q_{2n} = \dfrac{\gamma - c + \alpha\gamma}{2(1+\alpha)} \end{array}\right. \quad \text{if} \quad c \le \gamma(1+\alpha)$$

$$\text{and} \quad \begin{array}{l} q_{1n} = \dfrac{1-c+\alpha}{2(1+\alpha)} \\ q_{2n} = 0 \end{array} \quad \text{if } c > \gamma(1+\alpha).$$

In the second period, the retailer chooses quantity to maximize $\pi_2^{ret} = (p_{2n} - w_2)q_{2n}$. Taking first order conditions, we find the retailer's optimal response to w_2 to be

$$q_{2n}(w_2) = \frac{\gamma - w_2 + \alpha\gamma}{2(1+\alpha)}.$$

In the first period, the profits of the retailer are $\pi_1^{ret} = (p_{1n} - w_1)q_{1n}$. When used goods are not sold in the second period, there is no inter-temporal link between first and second period quantity choices due to the fact that there is a renewable market of consumers. Therefore, the retailer's optimal quantity choice in period 1 depends only on w_1. The best response of the retailer is given by:

$$q_{1n}(w_1) = \frac{1 - w_1 + \alpha}{2(1+\alpha)}.$$

The manufacturer can induce the retailer to choose the channel optimizing quantities in each period by offering a wholesale price equal to the marginal cost of production. In the second period, the retailer does not have the outside option to sell used goods. The manufacturer can use a fixed fee in each period to leave the retailer with profit just equal to its reservation payoff, which is normalized to zero without loss of generality. Hence the fixed fee in each period is equal to the period's channel profit. The resulting quantities are presented in Table 3 in the paper and the resulting manufacturer profit is presented in Table 4..

Proposition 3: *When the equilibrium solution involves sales of both new and used goods in the second period, the manufacturer and the channel earn greater profits than when there is no secondary market.*

Proof:

The retailer will sell new and used goods in the second period if $c < c^*(\gamma, \alpha, \theta)$.
The manufacturer's profit when there is an imperfect secondary market, from equation (12), is:

$$\pi^{mfgr} = \frac{(1+\alpha)^2(1+\gamma^2) - 2c(1+\alpha)(1+\gamma)}{4(1+\alpha)} + \frac{c^2(2-\theta)(2\alpha+\theta)}{4(1+\alpha)(2\alpha+\theta-\alpha\theta-\theta^2)} - \underline{\pi}^{ret}.$$

The manufacturer's profit when there is not an imperfect secondary market, from Table 4, is:

$$\frac{2c^2 - 2c(1+\alpha)(1+\gamma) + (1+\alpha)^2(1+\gamma^2)}{4(1+\alpha)} - \underline{\pi}^{ret}.$$

The difference between the manufacturer's profit when a secondary market exists and when there is no secondary market is given by:

$\dfrac{c^2\theta^2}{4(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)}$. This value is greater than zero for all $\alpha < \theta < 1$.

Q.E.D.

For completeness, we analyze whether this result holds when the retailer, in equilibrium, sells only used goods in the second period.

First, we look at when there are no new good sales in the second period when there is not a secondary market, $c \geq \gamma(1+\alpha)$:

The profit without a secondary market is $\dfrac{(1-c+\alpha)^2}{4(1+\alpha)} - \underline{\pi}^{ret}$. We are able to show that in this case, the manufacturer's profit is greater with a secondary market than without one.

If $\max\{\gamma(1+\alpha), c^*(\gamma,\alpha,\theta)\} < c < \overline{c}(\gamma,\alpha,\theta)$:

The manufacturer's profit when there is an imperfect secondary market, from Table 4, is:

$$\pi^{mfgr} = \frac{(\alpha(-1+\gamma)+\gamma\theta)^2}{4(2\alpha+\theta)} + \frac{4(\alpha(2-2c+\alpha+\alpha\gamma)+\theta(1-c+\alpha+\alpha\gamma))^2}{4(2\alpha+\theta)(4\theta(1+\alpha)+8\alpha+5\alpha^2)} - \underline{\pi}^{ret}.$$

We define $\Omega(c,\gamma,\alpha,\theta)$ as the difference between the profits when there is a secondary market and when there is no secondary market. We evaluate this difference at the upper bound on c for this case, $c = \overline{c}(\gamma,\alpha,\theta)$. This equals

$$\frac{(-\alpha+\gamma(\alpha+\theta))^2(3\alpha^3(64+37\alpha)+8\alpha^2\theta(32+23\alpha)+16\alpha\theta^2(7+6\alpha)+16\theta^3(1+\alpha)}{16(1+\alpha)(2\alpha+\theta)^2(3\alpha+2\theta)^2} > 0.$$

Note that:

$$\frac{\partial\Omega(c,\gamma,\alpha,\theta)}{\partial c}=\frac{\alpha(\alpha(1+3c+\alpha-4\gamma-4\alpha\gamma)-4\gamma\theta(1+\alpha))}{2(1+\alpha)(5\alpha^2+8\alpha+4\theta+4\alpha\theta)}.$$

Claim: $\frac{\partial\Omega(c,\gamma,\alpha,\theta)}{\partial c}$ is negative in the range for which there will be positive sales in the no secondary market case.

Proof: This derivative is clearly decreasing in γ. $\frac{\partial\Omega(c,\gamma,\alpha,\theta)}{\partial c}$ evaluated at the lower bound on $\gamma=\frac{\alpha}{\alpha+\theta}$ is equal to $-\frac{3\alpha^2(1-c+\alpha)}{2(1+\alpha)(\alpha(8+5\alpha)+4(1+\alpha)\theta)}$, which is negative for all $c<1+\alpha$. There are positive sales in the no secondary market case only if $c<1+\alpha$. Therefore, $\frac{\partial\Omega(c,\gamma,\alpha,\theta)}{\partial c}$ is negative for all *c* in the range.

Because $\Omega(c,\gamma,\alpha,\theta)$ is decreasing in *c* in the range of positive sales and positive at $c=\bar{c}(\gamma,\alpha,\theta)$, $\Omega(c,\gamma,\alpha,\theta)$ is positive for all $\max\{\gamma(1+\alpha),c^*(\gamma,\alpha,\theta)\}<c<\bar{c}(\gamma,\alpha,\theta)$.

If $\bar{c}(\gamma,\alpha,\theta)\le c\le\tilde{c}(\gamma,\alpha,\theta)$:

The manufacturer's profit when there is an imperfect secondary market, from Table 4, is:

$$\frac{2(1-c+\gamma\alpha+\gamma\theta)\left[\alpha\left(2-2c+\alpha+\alpha\gamma\right)+\theta\left(1-c+\alpha+\alpha\gamma\right)\right]}{\left[4\theta\left(1+\alpha\right)+8\alpha+5\alpha^2\right]}$$
$$-\frac{4(1+\alpha+\theta)\left[\alpha\left(2-2c+\alpha+\alpha\gamma\right)+\theta\left(1-c+\alpha+\alpha\gamma\right)\right]^2}{\left[4\theta\left(1+\alpha\right)+8\alpha+5\alpha^2\right]^2}-\underline{\pi}^{ret}.$$

We may define $\mathrm{T}(c,\gamma,\alpha,\theta)$ as the difference between the profits when there is a secondary market and when there is no secondary market. Evaluated at the lower bound on *c*, $c=\bar{c}(\gamma,\alpha,\theta)$, $\mathrm{T}(c,\gamma,\alpha,\theta)$ equals

$$\frac{(-\alpha+\gamma(\alpha+\theta))^2(3\alpha^3(64+37\alpha)+8\alpha^2\theta(32+23\alpha)+16\alpha\theta^2(7+6\alpha)+16\theta^3(1+\alpha))}{16(1+\alpha)(2\alpha+\theta)^2(3\alpha+2\theta)^2}>0.$$

Evaluated at the upper bound on *c*, $c=\tilde{c}(\gamma,\alpha,\theta)$, $\mathrm{T}(c,\gamma,\alpha,\theta)$ equals

$$\frac{(12\alpha+3\alpha^2+4\theta+4\alpha\theta)(-\alpha+\gamma(\alpha+\theta))^2}{4(1+\alpha)(3\alpha+2\theta)^2}>0.$$

Note that

$\frac{\partial \mathrm{T}(c,\gamma,\alpha,\theta)}{\partial c} = \frac{1-c+\alpha}{2+2\alpha} + \frac{\alpha-\gamma(\alpha+\theta)}{3\alpha+2\theta}$. $\frac{\partial \mathrm{T}(c,\gamma,\alpha,\theta)}{\partial c}$ is decreasing in c and γ. At the minimum value of c, $\frac{\partial \mathrm{T}(c,\gamma,\alpha,\theta)}{\partial c}$ is equal to $-\frac{\alpha(9\alpha+4\theta)(\gamma\alpha+\gamma\theta-\alpha)}{4(1+\alpha)(2\alpha+\theta)(3\alpha+2\theta)}$ which is negative for all $\frac{\alpha}{\alpha+\theta} < \gamma < \frac{2\alpha+\theta}{(2-\theta)(\alpha+\theta)}$. Therefore, $\frac{\partial \mathrm{T}(c,\gamma,\alpha,\theta)}{\partial c}$ is negative for all $\bar{c}(\gamma,\alpha,\theta) \le c \le \tilde{c}(\gamma,\alpha,\theta)$.

Thus, $\mathrm{T}(c,\gamma,\alpha,\theta)$ is positive over $\bar{c}(\gamma,\alpha,\theta) \le c \le \tilde{c}(\gamma,\alpha,\theta)$, because $\mathrm{T}(c,\gamma,\alpha,\theta)$ is quadratic in c, positive at $c=\tilde{c}(\gamma,\alpha,\theta)$ and $c=\bar{c}(\gamma,\alpha,\theta)$, and decreasing in c over this range.

If $\tilde{c}(\gamma,\alpha,\theta) < c < (1+\alpha)$*:*

The manufacturer's profit when there is an imperfect secondary market, from Table 4, is:

$$\frac{(1-c+\alpha\gamma+\theta\gamma)^2}{4(1+\alpha+\theta)} - \underline{\pi}^{ret}.$$

We may define $\Delta(c,\gamma,\alpha,\theta)$ as the difference between the profits when there is a secondary market and when there is no secondary market. Evaluated at the lower bound on c, $c=\tilde{c}(\gamma,\alpha,\theta)$, $\Delta(c,\gamma,\alpha,\theta)$ equals

$$\frac{(3\alpha(4+\alpha)+4\theta(1+\alpha))(-\alpha+\gamma(\alpha+\theta))^2}{4(1+\alpha)(3\alpha+2\theta)^2} > 0.$$

When $\Delta(c,\gamma,\alpha,\theta)$ is evaluated at the upper bound on c for there to be any good sold without a secondary market, $c=(1+\alpha)$, it is equal to

$$\frac{(-\alpha+\gamma(\alpha+\theta))^2}{4(1+\alpha+\theta)} > 0.$$

Note that

$\frac{\partial \Delta(c,\gamma,\alpha,\theta)}{\partial c} = \frac{1}{2}(1-\gamma-\frac{c}{1+\alpha}+\frac{-1+c+\gamma}{1+\alpha+\theta})$, which is negative for all $\tilde{c}(\gamma,\alpha,\theta) < c < 1+\alpha$, assuming $\frac{\alpha}{\alpha+\theta} < \gamma < \frac{2\alpha+\theta}{(2-\theta)(\alpha+\theta)}$. Thus, $\Delta(c,\gamma,\alpha,\theta)$ is positive over $\tilde{c}(\gamma,\alpha,\theta) < c < 1+\alpha$, because $\Delta(c,\gamma,\alpha,\theta)$ is quadratic in c, positive at $c=\tilde{c}(\gamma,\alpha,\theta)$ and $c=(1+\alpha)$, and decreasing in c.

Therefore, if $c > \gamma(1+\alpha)$, the manufacturer earns greater profits when there is an imperfect secondary market operated by the retailer than when there is no secondary market.

Finally, we look at when there are new good sales in the second period when there is not a secondary market, $c < \gamma(1+\alpha)$:

The profit without a secondary market is $\dfrac{2c^2 - 2c(1+\alpha)(1+\gamma) + (1+\alpha)^2(1+\gamma^2)}{4(1+\alpha)} - \underline{\pi}^{ret}$.

In this case, the manufacturer does not always earn greater profits with a secondary market than without one.

Comparing the profits of the manufacturer for $c^*(\gamma,\alpha,\theta) < c < \bar{c}(\gamma,\alpha,\theta)$ when the retailer operates a secondary market (Table 4) to the profits when there is not a secondary market, we find the manufacturer earns greater profits when the retailer does not operate a secondary market in the region:

$c^*(\gamma,\alpha,\theta) < c < c''(\gamma,\alpha,\theta) \le \bar{c}(\gamma,\alpha,\theta) \le \gamma(1+\alpha)$ where

$$c''(\gamma,\alpha,\theta) = \frac{(1+\alpha)(2\alpha+\theta)(\alpha^2+8\gamma+\alpha\gamma+4\gamma\theta)}{2(2\alpha+\theta)(\alpha^2+4\alpha+2\theta+2\alpha\theta)} -$$
$$\frac{\sqrt{(1+\alpha)(2\alpha+\theta)(5\alpha^2+8\alpha+4\theta+4\alpha\theta)(-2\alpha^3(\gamma-1)^2+\alpha^2\theta(\gamma-1)(1+\alpha-\gamma+3\alpha\gamma)+2\alpha\gamma^2\theta^2+4\gamma^2\theta^3(1+\alpha)}}{2(2\alpha+\theta)(\alpha^2+4\alpha+2\theta+2\alpha\theta)}.$$

Comparing the profits of the manufacturer for $\bar{c}(\gamma,\alpha,\theta) \le c \le \tilde{c}(\gamma,\alpha,\theta)$ when the retailer operates a secondary market (Table 4) to the when there is not a secondary market, we find the manufacturer earns greater profits when the retailer does not operate a secondary market in the region:

$\bar{c}(\gamma,\alpha,\theta) \le c < c'''(\gamma,\alpha,\theta) \le \tilde{c}(\gamma,\alpha,\theta) \le \gamma(1+\alpha)$ where

$$c'''(\gamma,\alpha,\theta) = \frac{(1+\alpha)(3\alpha+2\theta)(5\alpha+\gamma\alpha+2\theta)}{2(3\alpha+2\theta)^2} -$$
$$\sqrt{-(1+\alpha)(3\alpha+2\theta)^2(\alpha^2(25+\alpha)(\gamma-1)^2-4\alpha\theta(\gamma-1)(5-10\gamma+3\alpha+4\gamma\alpha)-4\theta^2(-1-\alpha+4\gamma-4\gamma^2+6\gamma^2\alpha)-8\gamma^2\theta^3}.$$

Comparing the profits of the manufacturer for $\tilde{c}(\gamma,\alpha,\theta) < c < 1+\gamma(1+\alpha)$ when the retailer operates a secondary market (Table 4) to the when there is not a secondary market, we find the manufacturer earns greater profits when the retailer does not operate a secondary market in the region:

$\tilde{c}(\gamma,\alpha,\theta) < c < \dot{c}(\gamma,\alpha,\theta) \le \gamma(1+\alpha)$, where

$$\dot{c}(\gamma,\alpha,\theta) = \frac{\gamma+\theta+\alpha(1+\alpha+\gamma+\theta)}{1+\alpha+2\theta} - \frac{\sqrt{(1+\alpha)(\alpha(\gamma-1)(2-2\gamma+\gamma\theta+\theta)+\theta(-1+4\gamma+\gamma^2(-3+2\theta)))}}{1+\alpha+2\theta}.$$

Proposition 4: *When the equilibrium solution involves sales of both new and used goods in the second period, the existence of a secondary market expands the number of consumers who ever buy the product. Specifically, sales of new goods in period 1 unambiguously increase with the existence of a secondary market; sales of new goods in period 2 decrease; and total new-good sales across the two periods may increase or decrease with a secondary market. If total new-good sales across the two periods decrease with a secondary market, incremental unit sales of used goods in period 2 more than compensate for the loss in new-good unit sales.*

Proof:

First, *the existence of a secondary market expands the number of consumers who ever buy the product* :

The retailer will sell new and used goods in the second period if $c < c^*(\gamma,\alpha,\theta)$.
In this range of *c*, the total quantity sold to consumers when there does not exist a secondary market is:

$$q_{1n} + q_{2n} = \frac{1-c+\alpha}{2(1+\alpha)} + \frac{\gamma-c+\alpha\gamma}{2(1+\alpha)} = \frac{(1+\alpha)(1+\gamma)-2c}{2(1+\alpha)} .$$

The total quantity sold to consumers when there exists a secondary market is:

$$q_{1n} + q_{2n} + q_{2u} = \frac{1-c+\alpha+\alpha\gamma+\gamma}{2(1+\alpha)} - \frac{2c\alpha(1-\theta)}{2(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)} .$$

The difference between total quantity with the secondary market and without is:

$$\frac{c\theta(1+\alpha-\theta)}{2(1+\alpha)(2\alpha-\alpha\theta+\theta-\theta^2)} > 0.$$

Therefore, when the retailer, in equilibrium, sells both new and used goods in the second period, the existence of a secondary market expands the number of consumers who ever buy the product.

Secondly, *sales of new goods in period 1 unambiguously increase with the existence of a secondary market.*

The difference between the first period quantity with the secondary market and the first period quantity without a secondary market is $\frac{\theta(1-\theta+c(\alpha+\theta-1))}{2\alpha(1+\alpha)(2-\theta)}$. When *c=0* this value is equal to $\frac{\theta(1-\theta)}{2\alpha(1+\alpha)(2-\theta)}$>0. If $c=1+\alpha$, which is the maximum value for which there will be new goods sold without a secondary market, this value is equal to $\frac{\alpha(\alpha+\theta)}{2\alpha(1+\alpha)(2-\theta)}$>0.

Therefore, this value is positive and when the retailer, in equilibrium, sells both new and used goods in the second period, sales of new goods in period 1 unambiguously increase with the existence of a secondary market.

Thirdly, *sales of new goods in period 2 decrease.*

The difference between the second period new good quantity with the secondary market and without the secondary market is equal to $\frac{c}{2(1+\alpha)}(1-\frac{2\alpha+\theta}{2\alpha+\theta-\alpha\theta-\theta^2})<0$.

Therefore, when the retailer, in equilibrium, sells both new and used goods in the second period, the sales of new goods in period 2 decrease with the existence of a secondary market.

Finally, *total new-good sales across the two periods may increase or decrease with a secondary market.*

The difference between the total new good quantity with the secondary market and without the secondary market is equal to $-\frac{\theta(1-2c-\theta+c\theta)}{2(\alpha+\alpha^2)(-2+\theta)}$. When $c<\frac{1-\theta}{2-\theta}$, this value is positive meaning the sales of new goods across the two periods increases with the existence of a secondary market. For $c^*(\gamma,\alpha,\theta)>c>\frac{1-\theta}{2-\theta}$, the total new-good sales across the two periods decreases with the existence of a secondary market. This range exists if $\gamma>\frac{(1-\theta)(2\alpha+\theta)}{(1+\alpha)(2-\theta)(2\alpha-\alpha\theta+\theta-\theta^2)}$.

Therefore, when the retailer, in equilibrium, sells both new and used goods in the second period, the total new-good sales across the two periods may increase or decrease with a secondary market.

□

For completeness, we analyze the impact of the secondary market on total sales in the other regions.

Region 1: $(q_{2u} - q_{1n}) > 0$, $q_{2n} = 0$ *with a secondary market and* $q_{2n} > 0$ *without a secondary market;* $c^*(\gamma,\alpha,\theta) \le c < \overline{c}(\gamma,\alpha,\theta)$ and $c < \gamma(1+\alpha)$.

The total quantity sold to consumers when there exists a secondary market is:

$$q_{1n} + q_{2n} + q_{2u} =$$
$$\frac{2\alpha(2-2c+\alpha+\alpha\gamma)+2\theta(1-c+\alpha+\alpha\gamma)}{5\alpha^2+8\alpha+4\theta+4\alpha\theta} + \frac{1}{6}(3\gamma - \frac{\alpha(1+\gamma)}{2\alpha+\theta} - \frac{2\alpha(1+3c+\alpha+\gamma(4+\alpha)}{5\alpha^2+8\alpha+4\theta+4\alpha\theta}).$$

The difference between the quantity sold with the secondary market and quantity sold without one is equal to $\frac{1}{6}(\frac{3c+3\alpha\gamma}{1+\alpha} - \frac{\alpha(1+\gamma)}{2\alpha+\theta} - \frac{\alpha(2+5\alpha)(1+3c+\alpha+4\gamma+4\alpha\gamma)}{(1+\alpha)(5\alpha^2+8\alpha+4\theta+4\alpha\theta)}$.
This is equal to zero at
$c = \frac{\alpha(1+\alpha)(5\alpha^2(1-\gamma)+2\theta+4\gamma\theta(1-\theta)+\alpha(4+3\theta+2\gamma(4-5\theta)))}{2(2\alpha+\theta)(3\alpha+2\theta+2\alpha\theta)}$ and positive for all greater values of *c*, by fact that its derivative with respect to *c* is:

$$\frac{3\alpha+2\alpha\theta+2\theta}{(1+\alpha)(5\alpha^2+8\alpha+4\theta+4\alpha\theta)} > 0.$$

In other words, for $c^*(\gamma,\alpha,\theta) \le c < \overline{c}(\gamma,\alpha,\theta)$, there exists a value of *c*, above which there is a greater total quantity sold when there is a secondary market operated by the retailer than when there is not.

Region 2a: $(q_{2u} - q_{1n}) = 0$, $q_{2n} = 0$ *with a secondary market and* $q_{2n} > 0$ *without a secondary market;* $\overline{c}(\gamma,\alpha,\theta) \le c \le \tilde{c}(\gamma,\alpha,\theta)$ and $c < \gamma(1+\alpha)$.

The total quantity sold to consumers when there exists a secondary market is:

$$q_{1n} + q_{2n} + q_{2u} = \frac{2(-\alpha+\gamma(\alpha+\theta))}{3\alpha+2\theta}.$$
The difference between the quantity sold with a secondary market and quantity sold without one is equal to $\frac{c}{1+\alpha} - \frac{\alpha(7-\gamma)+2\theta(1-\gamma)}{6\alpha+4\theta}$ which is positive over the relevant range of *c:* $\overline{c}(\gamma,\alpha,\theta) \le c \le \tilde{c}(\gamma,\alpha,\theta)$.

In this range, the existence of a secondary market expands the number of consumers who ever buy the product.

Region 2b: $(q_{2u} - q_{1n}) = 0$, $q_{2n} = 0$ *with a secondary market and* $q_{2n} > 0$ *without a secondary market;* $\tilde{c}(\gamma,\alpha,\theta) < c < \gamma(1+\alpha)$.

The total quantity sold to consumers when there exists a secondary market is:

$$q_{1n} + q_{2n} + q_{2u} = \frac{1 - c + \gamma(\alpha + \theta)}{1 + \alpha + \theta}.$$

The difference between the quantity sold with a secondary market and quantity sold without one is equal to $\frac{(1-\alpha^2)(1-\gamma) + \theta(2c + (1+\alpha)(\gamma - 1))}{2(1+\alpha)(1+\alpha+\theta)}$.

This difference is positive for over the relevant range of *c*: $\tilde{c}(\gamma, \alpha, \theta) < c < \gamma(1+\alpha)$.
In this range the existence of a secondary market expands the number of consumers who ever buy the product.

Region 3: $(q_{2u} - q_{1n}) > 0$, $q_{2n} = 0$ *with a secondary market and* $q_{2n} = 0$ *without a secondary market;* $\gamma(1+\alpha) \le c < \bar{c}(\gamma, \alpha, \theta)$.

The total quantity sold when there is not a secondary market and $c \ge \gamma(1+\alpha)$ is equal to:

$$q_{1n} + q_{2n} = \frac{1 - c + \alpha}{2 + 2\alpha}.$$

The difference between the quantity sold with a secondary market and quantity sold without one is equal to:

$$\frac{1}{6}\left(\frac{3\gamma(1+2\alpha)}{1+\alpha} - \frac{\alpha(1+\gamma)}{2\alpha+\theta} - \frac{\alpha(2+5\alpha)(1+3c+\alpha+4\gamma+4\alpha\gamma)}{(1+\alpha)(5\alpha^2+8\alpha+4\theta+4\alpha\theta)}\right.$$

This is positive for over the range of *c* for which $q_{2n} = 0$ without a secondary market; $\gamma(1+c) \le c < 1+\alpha$. Therefore, if $\gamma(1+\alpha) \le c < \bar{c}(\gamma, \alpha, \theta)$, the existence of a secondary market expands the number of consumers who ever buy the product.

In equilibrium, when there is not a secondary market, the total quantity sold when $c \ge \gamma(1+\alpha)$ is strictly less than the total quantity sold when $c < \gamma(1+\alpha)$ (The former implying $q_{2n} = 0$ and the latter implying $q_{2n} > 0$).
Because the existence of a secondary market expands the number of consumers who ever buy the product for Regions 2a, 2b and 3, we may conclude that the same result holds in the following two cases:

Region 4a: $(q_{2u} - q_{1n}) = 0$, $q_{2n} = 0$ *with a secondary market and* $q_{2n} = 0$ *without a secondary market;* $\bar{c}(\gamma, \alpha, \theta) \le c \le \tilde{c}(\gamma, \alpha, \theta)$ and $c \ge \gamma(1+\alpha)$.

Region 4b: $(q_{2u} - q_{1n}) = 0$, $q_{2n} = 0$ *with a secondary market and* $q_{2n} = 0$ *without a secondary market;* $\tilde{c}(\gamma, \alpha, \theta) < c < (1+\alpha)$ *and* $c \geq \gamma(1+\alpha)$.

Therefore, the secondary market expands the number of consumers who ever buy the product for any $c \geq \gamma(1+\alpha)$.
□

Q.E.D.

Proof that allowing for used used -oods to offer lower value to consumers than a used new-good does not have a qualitative impact on the results in this paper. (See Footnote 8)

We modify the model in the following way:

Second-period buyers' gross valuations of the goods are the same as for first period consumers, with the additional option of purchasing a used good:

$$V(\phi_2) = \begin{cases} (1+\alpha)\phi_2 & \text{if a new product is owned in the second period and subsequently} \\ \theta(1+\alpha)\phi_2 & \text{if a used product is owned in the second period and subsequently.} \end{cases}$$

By the same analysis as conducted in the paper (see Section 1.2), this leads to inverse demand equations:

$$\begin{aligned} c_u &= (1 - q_{1n} + q_{2u})\alpha \\ p_{1n} &= (1 - q_{1n}) + E(c_u) \\ &= (1+\alpha)(1-q_{1n}) + \alpha E(q_{2u}). \\ p_{2u} &= \theta(\gamma - q_{2n} - q_{2u})(1+\alpha) \\ p_{2n} &= (\gamma - q_{2n})(1-\theta)(1+\alpha) + p_{2u} \\ &= (1+\alpha)(\gamma - q_{2n} - \theta q_{2u}) \end{aligned}.$$

With this new model set-up we will show two things:

1) The profits of the integrated channel with new and used goods sold in the second period is greater than the profits of the integrated channel without a secondary market. (As shown in the paper, when new and used goods are sold, in equilibrium, in the decentralized channel, the manufacturer can replicate this outcome)
2) With a secondary market, the new good quantity in the first period will increase, the second period new good quantity will decrease and the total quantity sold across two periods will increase.

The results of Propositions 1 and 2 in the paper follow straightforwardly using the analysis set forth in the paper and technical appendix.

The channel's objective is again given by:

$$\max_{q_{2n},q_{2u},q_{1n}} \pi^2_{channel} = (p_{2n}-c)q_{2n} + (p_{2u}-c_u)q_{2u} + (p_{1n}-c)q_{1n}$$
$$s.t.\ \{q_{2n},q_{2u},q_{1n}\} \ge 0,\ q_{1n} \ge q_{2u}.$$

Restricting our attention to when the vertically integrated seller with quantity commitments to consumers will sell new and used goods in the second period, the optimal quantity choices are:

$$q^*_{1n} = \frac{\alpha(1-c)-\theta(c-(1+\alpha)^2)-\theta^2(1+\alpha)(1-c+\alpha)}{2\alpha+2(1+\alpha)^2\theta-2(1+\alpha)^2\theta^2}$$

$$q^*_{2n} = \frac{1}{2}(\gamma - \frac{c(\alpha+\theta+\alpha\theta)}{\alpha(1+\alpha)+\theta(1+\alpha)^3-\theta^2(1+\alpha)^3})$$

$$q^*_{2u} = \frac{c(\alpha-\theta-\alpha\theta)}{2\alpha+2(1+\alpha)^2\theta-2(1+\alpha)^2\theta^2}.$$

Notice, that again we may define a $c^*(\gamma,\alpha,\theta)$ such that $c < c^*(\gamma,\alpha,\theta)$ will imply positive new good sales in the second period and c> $c^*(\gamma,\alpha,\theta)$ will imply that the vertically integrated seller with quantity commitments to consumers will cease new good sales in the second period.

This leads to a total profit equal to

$$\frac{1}{4}(-2c(1+\gamma)+(1+\alpha)(1+\gamma^2)+\frac{c^2(2+\alpha-\alpha\theta-\theta)(\alpha+\theta+\alpha\theta)}{2\alpha+2(1+\alpha)^2\theta-2(1+\alpha)^2\theta^2}.$$

We now compare this profit to the profit without a secondary market. Note that the demand equations for when there is not a secondary market are unchanged by altering the assumptions about the valuations of buying used goods. The additional profits from the existence of the secondary market is then:

$$\frac{c^2(\alpha\theta-\alpha+\theta)^2}{4\alpha(1+\alpha)+4\theta(1+\alpha)^3+4\theta^2(1+\alpha)^3}>0.$$

Thus, the qualitative result of Proposition 3 is preserved with the change in assumptions.

The additional sales in the first period with a secondary market versus when there is no secondary market are given by:

$$\frac{c\alpha(\alpha\theta-\alpha+\theta)}{2\alpha(1+\alpha)+2(1+\alpha)^3\theta-2(1+\alpha)^3\theta^2}>0.$$

The decrease in sales of new goods in the second period with a secondary market versus when there is no secondary market is given by:

$$-\frac{c\theta(\alpha\theta-\alpha+\theta)}{2\alpha+2(1+\alpha)^2\theta-2(1+\alpha)^2\theta^2}<0.$$

The additional total units sold across period with a secondary market versus when there is no secondary market are given by:

$$\frac{c(1-\alpha\theta-\theta+2\alpha)(\alpha\theta-\alpha+\theta)}{2\alpha(1+\alpha)+2(1+\alpha)^3\theta-2(1+\alpha)^3\theta^2}>0.$$

Thus, the qualitative results of Proposition 4 are preserved with the change in assumptions.

Q.E.D.