Approaching Musical Actions

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SO, AN IMPROVISATION HAS BEEN GOING ON for some time, but its impetus is dying out, at first in a good way, all getting more quiet in a nice contrast to what has gone before, but soon, in fact now, we need a new idea, of course (inescapably) related to what we have been playing already, but one that will have a fresh effect and that can carry us into a fertile territory that will in some way complement what has gone before. I gather up into my mind and intuition some threads that have been woven into everything else so far, and form a tentative image of some new pattern to weave, and I act. The act is the public manifestation of my inner representation of my projection of the music we have played onto the screen of the future. The other musicians respond to this new musical context with their own representations, projections and actions, in a spreading web of new musical relations, represented individually and to some extent variously in each musician, and manifested publicly in our shared acoustic space, which serves as our blackboard—the space in which we communicate to each other.

Is the picture so different for a composer? She is sitting in a room. Nice trees wave in the breeze outside. She is digesting her breakfast egg over a cup of coffee. On her desk is a pile of pages of musical score, some 128 pages of her new piece for orchestra. She is half-way through the second of three large sections in the piece. She runs through the piece so far composed in her mind, pausing here and there to chew over some bit or other and revise her mental representation of the flow of the piece. She tries to form an image of a good way to continue further on into her second section. She sees the glimmer of an idea. But, it won't work after all that loudness 30 measures earlier. She puts down her coffee and acts: she erases the brass from a 20-bar stretch, leaving only the strings. She sketches in an oboe part in the blank score. . . .

I have referred in an earlier paper to this "wildfire of the musical swerve and flow" as "a sort of playful path in time through a field of temporally invariant relations."[1] David Lewin's point of departure for the development of his transformational networks was that "since 'music' is something you *do*, and not just something you *perceive* (or understand), a theory of music can not be developed fully from a theory of musical perception."[2] And again, Lewin says: "'If I am *at* s and wish to get to t, what characteristic gesture . . . should I perform in order to arrive there?' The question generalizes: 'If I want to change Gestalt 1 into Gestalt 2 . . . , what sorts of admissible transformations in my space . . . will do the best job?' This attitude is . . . the attitude of someone *inside* the music, as idealized dancer and/or singer. No external observer (analyst, listener) is needed."[3]

Lewin's formulation talks about changing Gestalt 1 into Gestalt 2. Recent practice of transformational theory has focused on networks of pitch classes, the so-called Klumpenhouwer networks, or Knets, which themselves only transform individual pitch classes into other ones, and on the isographies that can obtain between such networks. The isographies are, typically, used analytically in a rather static fashion to associate small stretches of music represented as networks of a few pc, much as a traditional motivic analysis would associate two small motives which bear some resemblance to each other.

We could quibble with Lewin's basic formulation, too, as taking some given (or found) thing and changing it into some other like thing. It seems altogether too focused on things. It may be an advance to think, as Lewin does, in terms of the transformation from one thing to the other, but the underlying granulation is rather grating. We want to think of music as growing. The thingness of music might lie in the magic metamorphosis from one thing, the music up to now as represented mentally and realized acoustically and in the score by acts, to a new and larger thing which is quite different, voilà, hey presto. Lewin-things are typically the same size, so they do not grow, and the transformations are structure-preserving, which means we are not surprised by the new thing because it is not really different. Are we convinced by an argument that the succession of the two similar things (in time or even in some ordered representation space) produces some really new thing, the one-then-the-other? It is a kind of repetition, which has its place in music for sure, and one can take this pretty far—I have done so myself—but it can't be the whole story.[4]

I want to recontextualize toward a representation of music which is more temporal, more complex, and located firmly within musical action. We can just begin by thinking about mathematical action, which in fact does model Lewin's thing-to-thing transformational idea pretty well. You take one thing and transform it into another similar thing. The transformations form some algebraic entity such as a semigroup, monoid, group, etc. There is always an action involved in a Net, the action of the arrow labels on the node contents. Lewin, in *GMIT*, refers to the transformations he uses to label his arrows as elements of a semigroup; more properly they are elements of a monoid, due to the underlying definition of digraph in *GMIT*. The whole situation can be reconceptualized and redefined as I have in my recent *JMM* paper "Cool Tools," to include polysemic and noncommutative Nets as well as Lewin-nets proper. There is still always at least an action of the arrow labels on the node contents.[5]

Let S be a monoid and A a non-empty set. There is standard mathematical definition of a *right [left]* S*-act* as a mapping from $A \times S$ to A where each pair of elements (a, s) maps to as, where $a1 = a$ and $a(st) = (as)t$ for all a in A and s, t in S. Where the identity is missing from S, this is called a semigroup act or "S-act"; where the identity element is present in S, one term for the action is unitary S-act, that is, a monoidal act. Since every group is a monoid, this also serves to define group action.

> Definitions: A *right [left]* S*-act* is a mapping from $A \times S$ to A where each pair of elements (a, s) maps to as, where $a1 = a$ and $a(st) = (as)t$ for all a in A and s, t in S. Where the identity is missing from S, this is called a *semigroup act*; where the identity element is present in S, one term for the action is unitary S-act, that is, a *monoidal act*.

An S-act is also called an S-automaton. It is a machine. A *semiautomaton* is an automaton without outputs. It is modeled as an act over a monoid in a natural way. In this case, A is the set of *states*, and S is the *input monoid*. In this way, in the theory of S-acts, we might as well speak of semiautomata instead of S-acts.[6]

Clearly then, Lewin transformational networks are semiautomata. (They do not have output.) When I pointed this out in Rahn (1994), David Lewin emailed me that some factory in Japan had in fact used his transformational theory to set up the production system.[7] One hopes that the model was adjusted to produce output. The advantage of semigroups and monoids over groups as a general model for machines is that not all machines can run backwards. Indeed, if we want to model musical acts as taking place in irreversible time, we will need to escape groups and inhabit monoids.

Let's review the specific situation for Knets before escaping Knets. Of course there is the action of the arrow-label group on the node contents, but this is relatively uninteresting in Knets because the node contents, individual pcs, are so simple, with no internal structure. Can you imagine our Composer (let us call her Isobel) meditating on the pitch class B♭? Delving imaginatively into its internal structure so as to find a way to move to some other pitch class? I do not speak of its spectral evolution and so on, just its quality as a B♭ pitch. As Jimmy Durante is reputed to have said of B♭, "That's a *Good* Note," but as a Good Note, it resides securely within itself, without necessity of change, a model of Parmenidean Being: "Being is without beginning and without end, whole, unique, imperturbable, and complete."[8]

We need to follow up our idea of how to grow a piece larger from some representation of its earlier stage. For this, we need at least a representation that is complex enough to characterize a moment of music-'til-now. Approaching this ideal, we could try to use as data objects sets of pcs, orderings of pcs, or sets or orderings of more complex data sub-objects such as "notes," represented as n-tuples of dimensional values such as start time, pitch, and so on, or yet more complex entities. We will address some of these later on.

Knets themselves have enough structure within them to begin to be interesting as data objects—they are a more specific representation of a tune or harmony than the set of pcs that are their node contents, for example, since they assert a structure among the pcs. Isobel might revolve in her mind some Knet-representable structure of pcs so as to come up with a following or larger compositional motive or harmony—though the issue remains of what might motivate her choice of some particular new motivic instance among all those with similar structures. So, given a music-thing represented as a Knet, we need to define a way of getting from the initial Knet to another Knet as a mathematical action on Knets. That is, we need to construct a Net whose node-contents are Knets, a kind of recursive Net, as Klumpenhouwer and Lewin have themselves discussed.[9]

I will suppose for Nets in general,[10] not just for Knets or Lewin-nets, this Principle of Action on Nets:

> **In order to define an action by some algebraic entity *H* on a Net as a whole it suffices to define an action on an arbitrary edge of the Net, that is, an action of any element *h* of *H* on the content of the first node *x*, the content of the second node *y*, and on the label (or color) of the arrow *g* from the first node to the second node (see Example 1).**

$$h \cdot [x \xrightarrow{g} y] \;\Rightarrow\; h \cdot x \xrightarrow{h \cdot g} h \cdot y$$

EXAMPLE 1: ACTION OF *H* ON A NET WHOSE ARROWS ARE LABELED IN *G*

In Knets, arrow-labels form a group, T_n/T_nI. Groups may act on themselves by left multiplication, by right multiplication, or by conjugation. However, if we wish to preserve structure in the result, we need the action of the group on itself to be an isomorphism, that is, an automorphism. Recall that if *H* is a normal subgroup of *G*, it maps into itself as a set of elements under conjugation by any element of *G*. So, if *H* is any normal subgroup of *G*, then by the definitions, *G* acts by conjugation on *H* as automorphisms of *H*, permuting the elements of *H*. If $H = G$ (which is possible because *G* is a normal subgroup of itself), this action is called an inner automorphism of *G*.

Klumpenhouwer himself investigated this situation for structure-preserving actions under the term "network isomorphism," for the case that $H = G =$ the T/I group (isomorphic to D_{24}) and node contents are individual pcs. If the action on the node contents is known, the problem to be solved in general is finding the action on the arrow-label, as shown in the commutative diagram of Example 2. Example 3 shows the solution here, where the group $H = G = T/I$. This case is simply the (inner) automorphisms of the T/I group, so that the action on the arrow-labels is by conjugation, and $? = hgh^{-1}$. The bottom right corner element of the diagram in Example 2 is then $hg(x) = hgh^{-1}h(x)$, as shown in Example 3, with the action on group elements $h \cdot g$ collapsing to simple composition of group elements.

$$
\begin{tikzcd}
x \arrow[r, "g"] \arrow[d, "h"'] & g \cdot x \arrow[d, "h"] \\
h \cdot x \arrow[r, "? = h \cdot g"'] & h \cdot (g \cdot x) = \; ? \cdot (h \cdot x)
\end{tikzcd}
$$

EXAMPLE 2: THE NET ACTION COMMUTATIVE DIAGRAM—PROBLEM IN GENERAL

$$
\begin{tikzcd}
x \arrow[r, "g"] \arrow[d, "h"'] & g(x) \arrow[d, "h"] \\
h(x) \arrow[r, "hgh^{-1}"'] & hg(x) = hgh^{-1}h(x)
\end{tikzcd}
$$

EXAMPLE 3: SOLUTION FOR GROUPS $H = G$

The calculation of hgh^{-1} is straightforward if slightly tedious, for the four cases. Example 4 shows the four commutative action diagrams for the four cases within the "network isomorphism" action set up in Example 3.

A more interesting version of this action is the generalization to the action of the automorphism group whose elements are of the form $T_aM_p(x) \rightarrow px + a$, where each p is coprime to the modulus of some equal tempered system of pcs. For the case ETS = 12, I will refer to this as the T/M group, the familiar full group of TTOs. Both T/I and T are normal subgroups of the full T/M group. Isomorphic action by conjugation still works if the group of arrow labels is any normal subgroup of the group acting on the Nets. For example, let H be the T/M group and G, the arrow-label group, be the group of transpositions, T_nM_1, or the T/I group, that is, $T_nM_{1,11}$—or alternatively, of course, another full T/M group, $T_nM_{1,11,5,7}$. Note that the familiar pcs that are the node-content of Knets can themselves be construed as the abelian group of transpositions, T, which is isomorphic to $\mathbf{Z}_{12}$.

It would be interesting to explore T/M for arbitrary ETS, as the group structures alter considerably. For example, when the modulus is itself a prime, such as 19 or 53, every element is coprime to it and

$$\begin{tikzcd}
x \arrow[r, "T_a"] \arrow[d, "T_n"'] \arrow[dr, "T_{n+a}"] & x+a \arrow[d, "T_n"] \\
x+n \arrow[r, "T_a"'] & x+a+n
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "T_a\mathrm{I}"] \arrow[d, "T_n"'] \arrow[dr, "T_{n+a}\mathrm{I}"] & -x+a \arrow[d, "T_n"] \\
x+n \arrow[r, "T_{2n+a}\mathrm{I}"'] & -x+a+n
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "T_a"] \arrow[d, "T_n\mathrm{I}"'] \arrow[dr, "T_{n-a}\mathrm{I}"] & x+a \arrow[d, "T_n\mathrm{I}"] \\
-x+n \arrow[r, "T_{-a}"'] & -x-a+n
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "T_a\mathrm{I}"] \arrow[d, "T_n\mathrm{I}"'] \arrow[dr, "T_{n-a}\mathrm{I}"] & -x+a \arrow[d, "T_n\mathrm{I}"] \\
-x+n \arrow[r, "T_{2n-a}\mathrm{I}"'] & x-a+n
\end{tikzcd}$$

EXAMPLE 4: THE FOUR COMMUTATIVE ACTION DIAGRAMS FOR THE FOUR CASES WITHIN THE T/I GROUP

generates the entire group cyclically. The number 12 is in fact unusually rich—superabundant—in factors, and therefore $\mathbf{Z}_{12}$ is unusually poor in cyclic generators.

To define specifically the action of the T/M group on Nets with arrow labels in any of its normal subgroups, we have to solve the diagram in Example 2 for the question mark ? for the case in which g is some T_aM_p and h is some T_nM_q, where a and n are elements of $\mathbf{Z}_{12}$ and p and q are 1, 5, 7, or 11. We know this will be hgh^{-1} so we just have to calculate what that is.

The first step is to find h^{-1}. For the left inverse, we set $h^{-1}\ T_nM_q = T_0M_1$ and solve the equation to get $h^{-1} = q^{-1}(x - n)$ (Example 5). Solving for the left inverse gets the same, as it should since automorphisms form a group. This result only makes sense for q coprime to the modulus. In this case, $q^{-1} = q$ since $q^2 = 1$, and the formula reduces as shown in Example 5. Written in the form T_nM_q, the inverse of T_nM_q is $T_{-qn}M_q$.

$h^{-1} = q^{-1}\ (x - n) = q\ (x - n) = T_{-qn}M_q$

e.g., inverse of T_nM_5: $T_{-5n}M_5(5x + n) = x + 5n - 5n = x$,

EXAMPLE 5: INVERSE OF T_nM_q

$$
\begin{aligned}
hgh^{-1} &= T_nM_q\, T_aM_p\, T_{-nq}M_q\,(x)\\
&= T_nM_q\, T_aM_p\,(qx - qn)\\
&= T_nM_q\,(pqx - pqn + a)\\
&= pq^2x - pq^2n + qa + n \qquad \text{since } q^2 = 1\\
&= px - pn + n + qa\\
&= T_{-pn+n+qa}\, M_p
\end{aligned}
$$

EXAMPLE 6: CONJUGATION IN T_nM_p

The next step is to calculate the conjugation hgh^{-1}. Set $h = T_nM_q$ and $g = T_aM_p$. The calculations are shown in Example 6. Written in the form of T_nM_q, the conjugation of T_aM_p by $T_nM_q = T_{-pn+n+qa}M_p$. This solves the diagram of Example 2 for ? within the T/M group, T_nM_p, $p = 1, 5, 7, 11$, modulus = 12.

We can arrive at a full 4 by 4 matrix of specific conjugations within the T/M group by substituting all the combinations of ps and qs into the $T_{-pn+n+qa}M_p$ formula from Example 6. The matrix is given in Example 7. Each matrix entry is a solution for the ? of Example 2 for one combination of values for p and q. All this works for non-commutative Nets as well as Lewin-nets.[11]

In a sense, all of this is just bookkeeping. Isobel may be more focused on the node objects, or perhaps on the arrows of the Net. Once Isobel decides to act mathematically on some Net representation, acting on the objects entails corresponding alterations in the arrows, and vice versa.

Remember that this is all part of our quest for more complex representations that might work as part of a model of *musical* actions—Isobel's acts. To further this quest, we can generalize this action of the T/M group in several ways. First, as noted, it is valid for Knets, and for more general Nets which include polysemic and noncommutative Nets,

$$hgh^{-1} = T_nM_q\; T_aM_p\; T_{-nq}M_q$$

$p =$	1	5	7	11
$q = 1$	T_a	$T_{-4n+a}M_5$	$T_{-6n+a}M_7$	$T_{2n+a}M_{11}$
5	T_{5a}	$T_{-4n+5a}M_5$	$T_{-6n+5a}M_7$	$T_{2n+5a}M_{11}$
7	T_{7a}	$T_{-4n+7a}M_5$	$T_{-6n+7a}M_7$	$T_{2n+7a}M_{11}$
11	T_{-a}	$T_{-4n-a}M_5$	$T_{-6n-a}M_7$	$T_{2n-a}M_{11}$

EXAMPLE 7: CONJUGATIONS OF THE T/M GROUP

as discussed in my *JMM* paper "Cool Tools." Second, it is valid for any modulus, as noted—that is, any ETS. Third, we can generalize the action to more complex node-contents than single pcs, so long as the group has an interpretation in which it can act on the node contents. We defined such an action for node objects that are themselves Knets, above. It is even easier when the node objects have less structure than Nets. For example, we are familiar with the notion that an action on a set of pcs is defined in terms of the set of images of the individual pcs under the action on pcs. Similarly, the action on an ordering of pcs can be defined as the ordering of the images of the pcs under that action. Therefore there is an action on Lewin-nets proper, and in general on Nets, whose node contents are sets of pcs or orderings of pcs rather than individual pcs. An action on a set of Nets can be defined in terms of the set of images of the individual Nets under that other action. Since a chain is a Net, and a chain-hom-set as defined in "Cool Tools" is a set of chains that are subnets of some larger Net, that is, a set of Nets, we can define a T/M action on chain-hom-sets, so long as the node contents of each chain are themselves the pcs or sets of pcs or orderings of pcs, etc., acted on by the T/M group.

These serve also to illustrate a principle that constrains *recursion of action* or even more generally, *layered action*: there must be a *bottom level* in which node content is simple in the sense that the action at that level is defined on it directly. By definition of action on a Net given in Example 1, action on a Net is action on some data set *S* which is the node contents, and action on the algebraic entities labeling the arrows of the Net. At each level, all actors must act directly on the arrow labels of the arrows at that level, and at least indirectly on the underlying data set *S*. All the levels must be *consistent* in that their algebraic entities and data objects eventually "fall through" to the same bottom level.

Let's look briefly at the case of an action on linear orderings of pcs. This is of interest in that the orderings themselves can be interpreted in various useful ways. Of course the orderings may be syntactic, as in serialism, or temporal-linear, as in a representation of a motive. In such cases, conventionally, the action on the ordering is uniform—the same group element acts identically on each of the pcs in the ordering, as shown in Example 8.

$$g(\langle \mathrm{pc}_1, \mathrm{pc}_2, \ldots, \mathrm{pc}_n \rangle) = \langle g(\mathrm{pc}_1), g(\mathrm{pc}_2), \ldots, g(\mathrm{pc}_n) \rangle$$

EXAMPLE 8: HOMOGENEOUS UNIFORM ACTION ON ORDERINGS

This is a simple two-level action. It could form part of more complex actions, for example, as the next to the bottom and bottom levels of an action on Nets of sets of sets of Nets of orderings of orderings . . . of orderings of pcs.

But consider another action on orderings of pcs which is not uniform (Example 9). In this case, each pc at the bottom level is subject to a different operation within a larger action on the ordering of pcs. Of course, g would have to be defined as having components g_i which act in an appropriate fashion. In this case, the component actions are independent, so we can use the direct product of the underlying group acting on the pcs with itself n times, G^n where n is the number of elements in the orderings acted on.

$$g = \langle g_1, g_2, \ldots, g_n \rangle$$

$$\text{for } f, g, \text{ elements of } G^n,\ fg = \langle f_1 g_1, f_2 g_2, \ldots, f_n g_n \rangle$$

$$g(\langle \mathrm{pc}_1, \mathrm{pc}_2, \ldots, \mathrm{pc}_n \rangle) = \langle g_1(\mathrm{pc}_1), g_2(\mathrm{pc}_2), \ldots, g_n(\mathrm{pc}_n) \rangle$$

EXAMPLE 9: HOMOGENEOUS NON-UNIFORM ACTION ON ORDERINGS, $g \in G^n$

There has been considerable attention paid recently to transitions and relations between chords that are sorted into voices in some way, that is, abstractly, by register, by instrument, or whatever. Voice-sorted chords are properly represented as ordered n-tuples of pitches or pcs, so that (unlike in multisets) you can keep track of the voices.[12] For any of these problems one can use the action of Example 8, for G = T or T/I or T/M or some restriction of these such as moving at most one or two voices by 1 or 2 semitones within group T.[13]

Examples 8 and 9 are labeled as "*homogeneous uniform action*" and "*homogeneous non-uniform action.*" A *uniform action* on a complex object operates on each element of the complex object by the same element of the algebraic entity, e.g., the same group operation. In Example 8, each pc in the ordering pc_i is operated on by g. The *non-uniform* action of Example 9 operates on each element of the ordering pc_i by a different group element g_i.

A *homogeneous action* is a multilevel action in which the same algebraic entity is used at all levels, as is the case in Examples 8 and 9. A more complex example would be using the T/M group at each level of the "action on Nets of sets of sets of Nets of orderings of orderings . . . of orderings of pcs," with each action of the T/M group homogeneously "falling through" to the next level and eventually to the pcs at the bottom level. Clearly, a homogeneous action may be either uniform or non-uniform.

The idea of a *heterogeneous action* is a bit more complicated. It is possible that at different levels of a multilevel action, different algebraic entities are used to move to the next level. At each level, the action to the next level must be properly defined on the objects and arrows. Note that this action may be either uniform or non-uniform.

Let the action from the i-th to the i+1-th level be notated Act_i. The specification of each Act_i proceeds as usual; examples 8 and 9 are examples of such a specification for uniform and non-uniform actions, respectively. See Example 10. Level-hetero actions are layered, but not recursive.

> For action at n levels, define a set $\{\mathrm{Act}_i\}$ for $i = 1, \ldots n - 1$, such that each Act_i of the set $\{\mathrm{Act}_i\}$ takes level i into level $i + 1$.

EXAMPLE 10: HETEROGENOUS ACTION

There is one more distinction we can make among complex actions. Define a *note* (we could even use the term "sound," but this might get confused with the aural sensation) as an (ordered) n-tuple (list, vector) of dimensional values in a way familiar from computer music; for example <start-time, duration, pitch, loudness, usw>. This representation might be quite complex; I have written and used csound instruments that take over 40 parameters. However, for easiest mathematical treatment we would want to make sure that all the parameters were independent of one another. The case of parameters that are partly dependent on each other, for example the usual treatment of attack, duration, sustain and decay controls in computer music, is more complicated.

Suppose then we have a note with n independent parameters p_i, represented as a list or n-tuple. In general, each parameter may have a different perceptual space, which may require a different algebraic object—let's just say, group—to act on the objects in that space. If the parameters are independent, we can define an action on the notes using the direct product group of the respective groups for each parameter in the list. (See Example 11.) The action in Example 11 is non-uniform because each parameter is acted on by a different group element, but in addition, the groups from which the group elements are taken also vary from parameter to parameter. Instead of being an exponentiation of some one

$$h \in \text{direct product of possibly different groups, } H = G^1 \times G^2 \times \ldots \times G^n$$
$$h = \langle g_1, g_2, \ldots, g_n \rangle, g_i \in G^i$$
$$\text{for } f, g \text{ elements of } H, fg = \langle f_1g_1, f_2g_2, \ldots, f_ng_n \rangle$$
$$h(\langle p_1, p_2, \ldots, p_n \rangle) = \langle g_1(p_1), g_2(p_2), \ldots, g_n(p_n) \rangle$$

EXAMPLE 11: LEVEL-HOMOGENEOUS GROUP-HETEROGENEOUS NON-UNIFORM ACTION ON ORDERINGS

group G as in Example 9, this is the direct product group of a number of different groups. Yet this action remains homogeneous in the sense that the same action is applied at all its levels (one transition). We call it a "level-homogeneous group-heterogeneous non-uniform action."

So actions can be: level-homo or level-hetero, group-homo or group-hetero. If group hetero, then non-uniform. If group-homo, then either uniform or non-uniform. Ignoring "hard" dependencies such as "if group-hetero then non-uniform," there would be a total of eight such classifications, a sort of Eight-Fold Way. If we bring other entities such as rings, fields, and modules into the picture, the scheme exfoliates too madly to bother with.

Admittedly, this classification scheme for actions is rather complicated. However, this is one of its virtues, leading us away from any simplistic tendencies and toward ideas of mathematical actions that are complex enough perhaps to begin to model Isobel's musical acts.

Finally, let's take a different approach to Isobel's representations.[14] We informally tie together three notions: relation, net, and space. Define an n-ary relation R in the usual way as any subset of the Cartesian product of n sets. If all the sets in the Cartesian product are the same, the product is homogeneous; if not, heterogeneous. If it is homogeneous, we call it a relation "on" its unique underlying set. Considering R as a relation, we can ask if it is reflexive, symmetric, transitive, connected, and so on. This can also be considered an n-dimensional net, or digraph, when a net is defined as Lewin did originally, identifying arrows with ordered pairs (tuples, in the general case) of objects. This is also interpretable under certain conditions as the total note-space, with each point in it a note as defined above. Each subspace, and in particular each dimension, would have its own metric, defining its interval structure, and so on in the familiar development of transpositions and other isometries and isomorphisms.

We can color this net or relation in more or less elaborate ways by giving a formal *interpretation* of it, as defined in Example 12 for

An *interpretation* of a relation R on a set S is a semantic function $F_{sem}(x_i) \rightarrow w_i$ assigning each x_i in S an object w_i in the world W in such a way that for every ordered n-tuple $\langle x_i \rangle$ in R (for i ranging from 1 to n), the corresponding n-tuple of images $\langle F_{sem}(x_i) \rangle$ is in relation RW where RW is the "real world" relation being modeled by R.

EXAMPLE 12: DEFINITION OF AN INTERPRETATION

homogeneous relations on a set S (the definition for heterogeneous relations follows *mutatis mutandis*).

A simplified version of this idea of interpretation can serve to define the more general Net idea, by adding a coloring, an n+1th place to each n-tuple which is occupied by a list of arrow labels. (There is more to it than this, but I am leaving it out to simplify; for a fuller formal semantics, see my article "Network Models.") Each such list of arrows is $\mathrm{homdirect}_R(x, y)$, the list of all arrow-labels directly from point x to point y (path of length one). This would be an alternative to my construction in "Cool Tools."

A coloring can also be viewed as a binary relation, between the n-tuple points in the space (or if it is viewed as a graph, arcs) and the colors in the n+1th place. In general any n-ary relation can be built recursively from binary relations. The set of all homogeneous binary relations on a set Υ, that is, the power set of $\Upsilon \times \Upsilon$, with composition of binary relations defined in the usual way (see Example 13), is a monoid, with the diagonal of all and only $\langle x, x\rangle$ as its identity relation. Monoidal acts are defined for it (see the definition of S-act). So we could define a Net whose node contents are binary relations, and whose arrow labels are also binary relations that act on the node content! This would open up quite a new field of inquiry about Nets.

For binary relations R, S on a set X, the right [left] composition of R and S, $R \circ S$ =
$\{(x, z)$ in $X \times X \mid$ there exists y in X with $x\,S\,y$ and $y\,R\,z\}$

EXAMPLE 13: COMPOSITION OF BINARY RELATIONS

Finally, consider phrase-structure grammars, which have been used as models of Schenkerian-type theories of tonal music and therefore are one kind of plausible candidate for external representations of Isobel's internal representations of piece-'til-now and piece-to-come. Any such grammar is a kind of finite-state machine and therefore *can be formalized as an* S*-act.*

These grammars produce structures that are trees, which are a kind of graph or net (or relation!) that is partially ordered in a particular way. Note that the grammar does tie together in this way the ideas of mathematical action, relation, and net.

Example 14 is taken from my 1994 article, "Network Models." It shows a little invented phrase-structure grammar for a kind of non-tonal music, purely as a methodological illustration, formalized as a formal theory with axioms and with inference rules in the form of Emil-Post

S = all subsets of the set of integers mod 12
$P = \{P1, P2, P3, P4\}$ where
- $P1$: $\varnothing \rightarrow \{0\}$
- $P2$: $X \rightarrow T1(X)$ where $T1$ is transposition by 1
- $P3$: $X \rightarrow X\text{-}\{0, 2, 5\}$
- $P4$: $X\text{-}\{0, 2, 5\} \rightarrow \{0, 3, 7\}\text{-}X\text{-}\{0, 2, 5\}$

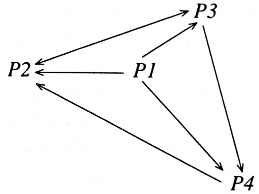
A derivation:

rule	*string*
axiom	$\varnothing$
P1	{0}
P3	{0}-{0, 2, 5}
P2	{0}-{1, 3, 6}
P3	{0}-{1, 3, 6}-{0, 2, 5}
P4	{0, 3, 7}-{0}-{1, 3, 6}-{0, 2, 5}
P2	{1, 4, 8}-{0}-{1, 3, 6}-{0, 2, 5}

string of productions: *P1-P3-P2-P3-P4-P2*

metagrammar (finite state):
- *P2* is followed by *P3*
- *P*3 is followed by *P4* or *P2*
- *P4* is followed by *P2*

diagram:



EXAMPLE 14: GRAMMAR AND METAGRAMMAR

productions, along with one particular derivation sequence modeling the structure in this grammar of one of the musical pieces, that is, theorems or sentences, producible by this grammar.

Imagine Isobel imagining her piece in this way, that is, a basic vocabulary of pc sets, a secondary vocabulary emphasizing pc sets of T_n-type (0 3 7) and (0 2 5) and emphasizing transformation of any pc set by T_1. She realizes that one stretch of her music, perhaps even the music-'til-now, is the particular production within this grammar shown in Example 14 as a derivation. Clearly, many other stretches of music can be represented as sentences within this grammar, but not just any stretch of any music. So the grammar as a whole lends a certain flavor to the piece, which from Isobel's standpoint, is good.

But what does Isobel do next? She would like to move to some closely related new stretch of music, but it should not be so closely related that it does not sound like something new. To generate just any new stretch from the grammar would probably not constrain the production enough, that is, the new thing would not necessarily be close enough to the earlier thing.

Isobel realizes that the *particular derivation sequence* that produced her piece-'til-now would constrain new production more than the grammar as a whole, but in a way that need not produce something too close. She can *theorize the production sequence*, as shown in the "meta-grammar" in Example 14, as itself a kind of finite-state machine. This meta-grammar is the bottom diagram of Example 14. Isobel can operate this machine to get a next slice of her piece that is closely, but not too closely related, to what has come before.

We have come to the end of this for now, leaving further development and applications of all these ideas to later research. I will feel successful if this has focused attention on the formal modeling of creative musical acts, and has illustrated and encouraged thinking in terms of models that are not simplistic, but are complex enough to be credible.

NOTES

This paper was first presented as a keynote address in Berlin, May 2007, at the first meeting of the Society for Mathematics and Computation in Music.

1. Rahn (2004), p. 136.
2. Lewin (1986), p. 377.
3. Lewin (1987), p. 159.
4. Rahn (1993).
5. Rahn (2007).
6. Kilp, Knauer, and Mikhalev (2000), pp. 43–5.
7. Rahn (1994), p. 232.
8. Parmenides from Rahn (2004), p. 131.
9. Lewin (1990). The discussion I develop below specifically about actions on Knets arrives at results consistent with those in Lewin's Appendix A and B in his article, which lay out the inner automorphisms of the T/I and T/M group, but my discussion focuses on (dynamic) actions, not (static) isomorphic relations, and uses a different line of argument.

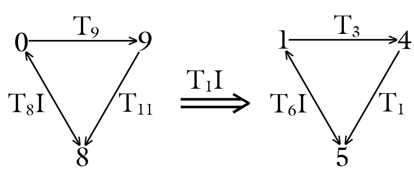
   The idea of conjugate action within the T/M group has a complex history in music theory. As far as I know, the specifics on conjugation within the T/M group, along with many other useful ideas, were first published by Daniel Starr, in his excellent article "Sets, Invariance, and Partitions" (Starr 1978). On p. 28, Starr develops a formula for conjugation with a result equivalent to my Example 6, in a different format. But Starr uses this to discuss the invariance of sets of pcs, focusing on the more usual action of the group on sets of pcs rather than the action of the group on itself. I had forgotten about Starr's earlier presentation, and am grateful to Robert Morris for reminding me of it. The demonstration and proofs here are independent.

   Robert Morris also presented the inner automorphisms of the TTOs in Morris (1987), p. 169. You would have to change the order of operations on Morris's table heads and rows, flip around the main diagonal (so rows interchange with columns), change names of operators and their arguments, and adjust the arithmetic in the subscripts, to get my Example 7. Again, my development here is independent of Morris's. See also Morris (2001).

It is possible that Starr's article also lurked, half-forgotten, in David Lewin, since the notation for the automorphisms in Lewin 1990 as "F<*u*,*j*>" is so close to the notation Starr uses (Starr 1978, p. 11 ff, and Table 4), which is "F = <*a*, *b*>" such that the action on pcs is $F(x) = ax + b$, with the multiplier written first and the transpositional subscript second in the ordered pair, like Lewin's notation. (Lewin cites Morris (1987), but not Starr (1978).) However, Lewin does not speak in terms of actions at all, which may have led him (seeking to represent a distinction inherent in the notion of action, without action) to cast the automorphisms as a different group than the group of TTOs, when actually it is all the same group, acting on itself by conjugation.

10. Nets generalize Lewin-nets, which generalize Knets. See the definitions in Rahn (2007).

11. For a non-commutative example, just take T_1I acting on the Net with pcs 0, 9, and 8 as nodes, and arrows T_9 from 0 to 9, T_{11} from 9 to 8, but T_8I from 0 to 8. T_9 and T_{11} do not compose to T_8I so this is not commutative, and is not properly a Knet at all (perhaps a "KNet"). Using the table in Example 7, T_1I acting on this Net maps contents 0 to 1, 9 to 4, and 8 to 5, and maps arrows T_9 to T_3, T_{11} to T_1, and T_8I to T_6I, as it should. Any additional arrows in a polysemic version would also check out.



Note that non-commutative Nets are a significant generalization, in that they make possible many analytical assertions in the form of Nets that are not possible when the Nets are constrained to be commutative, as are Knets. No longer are you restricted to the few, tightly constrained Knet arrangements of three-node Nets, for example, if you want to assert three-node Nets at all.

12. A multiset simply indexes its elements by their multiplicities, so a given content element can appear more than once, but the elements are still unordered; it still has no way of tracking a particular element position from set to set.

13. Brandon Derfler is writing a Ph.D. dissertation at the University of Washington on parsimonious voice-leading chord spaces using this idea, entitled "Single-Voice Transformations: A Model for Parsimonious Voice Leading."

14. What follows is indebted to Rahn (1994).

References

Kilp, Mati, Ulrich Knauer, and Alexander V. Mikhalev. 2000. *Monoids, Acts, and Categories, with Applications to Wreath Products and Graphs: A Handbook for Students and Researchers.* Berlin: de Gruyter.

Lewin, David. 1986. "Music Theory, Phenomenology, and Modes of Perception." *Music Perception* 3, no. 4 (Summer): 327–92.

———. 1987. *Generalized Musical Intervals and Transformations.* New Haven: Yale University Press.

———. 1990. "Klumpenhouwer Networks and Some Isographies That Involve Them." *Music Theory Spectrum* 12, no. 1 (Spring): 83–120.

Morris, Robert. 1987. *Composition with Pitch Classes.* New Haven: Yale University Press.

———. 2001. *Class Notes for Advanced Atonal Theory.* Lebanon N.H.: Frog Peak Music.

Rahn, John. 1993. "Repetition." *Contemporary Music Review* 7: 49–57.

———. 1994. "Some Remarks on Network Models for Music." In *Musical Transformation and Musical Intuition: Eleven Essays in Honor of David Lewin*, ed. Raphael E. Atlas and Michael Cherlin. Roxbury, Mass.: Ovenbird. 225–35.

———. 2004. "The Swerve and the Flow: Music's Relationship to Mathematics." *Perspectives of New Music* 42, no. 1 (Winter): 130–48.

———. 2007. "Cool Tools: Polysemic and Noncommutative Nets, Subchain Decompositions and Cross-Projecting Pre-Orders, Object-Graphs, Chain-Hom-Sets and Chain-Label-Hom-Sets, Forgetful Functors, Free Categories of a Net, and Ghosts." *Journal of Mathematics and Music* 1, no. 1 (March 2007): 7–22.

Starr, Daniel. 1978. "Sets, Invariance and Partitions." *Journal of Music Theory* 22, no. 1 (Spring): 1–42.