DEDICATED TO PROFESSOR Z. W. BIRNBAUM

RELIABILITY AND FAULT TREE ANALYSIS

Theoretical and Applied Aspects of System Reliability and Safety Assessment

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BIRNBAUM'S CONTRIBUTIONS TO RELIABILITY THEORY

SAM C. SAUNDERS*

Abstract. This paper contains a brief historical account of the career of Professor Z. W. Birnbaum with its relation to the foundation and development of Reliability Theory during the past two decades. It contains a synoptic description of the main contributions of his papers dealing with reliability or life length and provides references to the appropriate selections from his bibliography. This paper is adapted from the remarks at the session dedicated to Professor Birnbaum upon the occasion of his retirement.

1. Introduction.

It is easy enough to speak words of praise for persons who have either attained wealth or power, or who have the consideration to be dead. But whereas the subject of this discourse has not the courtesy to fulfill any of these usual prerequisites and by his very presence prohibits any artistic license with factual presentation, and moreover by being the next speaker can contradict and correct any unfavorable interpretation of events, these circumstances constitute a grave impediment to the usual and natural tenor of such declamations as this.

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In addition to these difficulties, I must remind you that Professor Birnbaum is not a man to whom everyone looks up.* In fact I estimate 85% of all the men and 50% of all the women working in Reliability Theory today look down on him. The sad fact is, he is not a giant among the workers in his field. What can I say about a man like that?

Well, I can report that he is a man of rectitude. On one of his consulting trips to the Los Angeles area, it is reported perhaps apocryphally, that some of his friends took him to a cocktail bar. It was done in an architectural style best described as "California Rococco" with the waitresses dressed in "minimally flamboyant" costumes. As he walked in Professor Birnbaum was heard to remark "Well, I will have one drink but I will not go upstairs".

I can report that he is a man of principle. He is not now, nor ever has been, a member of the Baysian Subversive Society. When the inferential problems become too difficult, he has never, to my knowledge, invoked the name of the Reverend Thomas Bayes as a talisman against evil fortune and then assumed an expedient prior distribution from within the conjugate class.

Those of you who do not know Professor Birnbaum can, from these anecdotal descriptions, recognize him as being short, upright and square.

*Prof. Birnbaum is 5'4" tall.

2. Professor Birnbaum was educated in mathematics by the Faculty of Science at the University of Lwow, Poland. He graduated with a Ph.D. in 1929 after having previously received a degree from the Faculty of Law in 1925. (He gave up Law, I understand, because he couldn't train himself not to keep his hands in his own pockets.) He studied under Professors Hugo Steinhaus, Stefan Banach and Kazimierz Kuratowski at that renown Polish School of Mathematics at Lwow, where was located the coffee shop with its famous problem book. His contemporaries were Stanislaw Ulam, Stefan Mazur, P. Schauder and Wladyslaw Orlicz, to name a few.

This education, in classical and functional analysis, accounts for his first dozen or so publications with such titles as "Zur Theorie der Schlichten Funktionen" in Studia Mathematica, see [25].

It is the current, conventional wisdom that anyone with the "advantage" of such an education, at such a school, with such classmates could not but be successful. Moreover, anyone admitted to such a program would enjoy a life of prosperity, happiness and satisfaction. This egalitarian philosophy would have you believe that if Birnbaum had been admitted to the Green Bay Packers training camp, under the tutelage of Vince Lombardi, he could have become a football linebacker like Ray Nitschke, while Ray Nitschke, pursuing a course of study under Steinhaus could have become a mathematician like Birnbaum.

Dr. Birnbaum spent two years, 1929-31, at the University of Göttingen, Germany - the Mecca in Applied Mathematics. At that time there were such professors as David Hilbert, Edmund Landau, Felix Bernstein and Emma Noether, along with such visitors as Andrie Kolmogorov and Paul Alexsandrov.

There Dr. Birnbaum studied Actuarial Science, among other subjects, and during the mid-thirties, 1931-36, he was employed as Chief Actuary for the Phoenix Life Insurance Company of Vienna. (It is my understanding that the company went into bankruptcy soon after Dr. Birnbaum's departure. It is now the Vienna Life Insurance Company of Phoenix.)

Noting the political developments on the continent at that time and in his own words "with an intimation of the coming unpleasantness", he immigrated to this country in 1937 and spent the first two years on a research appointment at New York University. Then in 1939 he accepted an appointment in Mathematics at the University of Washington, where he remained. This year he reached retirement, after completing 35 years of service to the Department of the University.

During those years he was made a Fellow of the American Statistical Association and a Fellow of the Institute of Mathematical Statistics. He was the National President of the Institute of Mathematical Statistics and served as Editor of the Annals of Mathematical Statistics. He has served on both the visiting lecturer programs of SIAM and of the Statistical Societies. During sabbatical years he was Visiting Professor at Stanford, the Sorbonne, and the University of Rome.

Professor Birnbaum is well known for his work in Distribution - Free Methods in Statistics. His contributions to this field constitute another dozen or two publications in the late forties and fifties. I recall at least five Ph.D. students who wrote theses under his direction in this field.

These papers need not concern us here except that I remember somebody saying "I was lucky to find the distribution of that statistic before Birnbaum could prove that it was distribution - free".

In addition to his academic attainments Dr. Birnbaum was for many years employed as a consultant to Industry and Government. These included The Boeing Company, Systems Development Corporation and the U.S. Department of Health, Education and Welfare.

The breadth of his mathematical knowledge, coupled with his depth of understanding, are the reasons for his being such a successful consultant in so many fields of applied statistics. Someone once remarked to me that he had never seen anyone present a problem to Professor Birnbaum but what he went immediately to the board and began a systematic attack upon it. I can testify to the truth of this and add that in my memory Professor Birnbaum always began writing with the same remark "Let me proceed slowly like a mule and write down all the things that you know so well, already". Whereupon he usually proceeded to write down some fundamental relationships that one did not know at all.

As a graduate student I held Professor Birnbaum in the same awe with which I held all Full Professors and persons who can leap tall buildings in a single bound. However, my admiration for him was not kindled until a few years later at a Pacific-Northwest SIAM Dinner meeting at which he was the featured speaker, he arose and said that before his prepared remarks he would like us all to join him in a demonstration of applied mathematics. He proposed we utilize the principle of Torricelli for the transport of alcoholic liquid through thin hollow tubes. Whereupon he proceeded to demonstrate the clear advantage of a European Education in such particulars.

3. Life Length Distributions. Dr. Birnbaum's knowledge of actuarial procedures made him an expert in the proper calculation of life contingency. This resulted in two publications before 1950, see [1] and [2], there he and his co-authors maintained that a little careful analysis of the extant data on the morphology of chemically induced cancer showed that the incidence of carcinoma does increase with age, contrary to the published opinion of others.

It is difficult today to understand the lack of acceptability, twenty five years ago, of what we presently think of as Reliability Theory. Actuarial Science, with its approach to life length calculations, was virtually regarded as a completed field. The fundamental work of Epstein and Sobel [26], assuming constant failure rate, was just being assimulated by the statistical community.

Professor Wallodi Weibull recounted to me that the now famous paper of his "A Statistical Distribution of Wide Applicability", in which was first advocated the "Weibull" distribution with its failure rate a power of time, was rejected by the Journal of the American Statistical Association as being of no interest. Thus one of the most influential papers in statistics of that decade was published in the Journal of Applied Mechanics. See [35]. (Maybe that is the reason it was so influential!)

In the early fifties it was presumed by most practitioners in Engineering that all non-deterministic data, including life length observations, should be analyzed through the use of the error function. If the data was too disperse to make this a reasonable procedure (it was not reasonable if, say, the estimated 10th percentile of life was a negative number) then one must first apply some transformation to make the data "normal".

Following this, one performed the usual statistical calculations of mean and variance and then re-transposed to determine the low-life percentiles. One of the favored transformations was the logarithm. But in some instances where fatigue life data were from structures, and hence more variable, it was found necessary to take the logarithm of the logarithms of the observations to reduce the scatter sufficiently. This ad hoc procedure was in fact recommended! See [20].

This was an unsatisfactory state of affairs, especially to anyone with actuarial training and experience plus an extensive knowledge of both analysis and statistical methodology. Consequently, (in a joint paper, see [3]), Professor Birnbaum undertook one of the first efforts to calculate the reliability of a structure with the imposed loading function as a parameter. In this paper the statistical problems of estimating parameters were considered to be of equal importance with the calculations of load made by a stress analyst. This was an innovative step because it required not only observations of the times of failure but of the imposed load histories as well.

The attempt in [3] was to develop a class of reliability distributions (survival functions) for materials in which the failure rate was the product of two factors; one due to the load, as determined by stress analysis, and the other due to aging alone, independent of the load including such influences as corrosion, oxidation or crystalline changes.

What we now call "a k out of n structure" was presumed built with such components and subjected to a load which was shared equally by all surviving components, such as would occur in a rope with a critical number of strands. The distribution for the life of this structure was derived and the model was applied to some fatigue data on metallic coupons subjected to sinusoidal loading. The modest claim of this life length model was that the concepts were intuitive, the resulting distributions manageable, procedures for estimation simple and the results yielded good agreement with available data.

More importantly these ideas could be extended to random loading functions. If the loading itself is a realization from a random environment, such as gust buffeting, then the failure rate itself becomes a stochastic process and the reliability becomes a stochastic integral. Such development was considered by Professor Birnbaum in [4] and [5]. If we consider a structure as a matrix each element of which is such a component, we have a generalization considered by Professor Richard Savage and his collaborators, see [19], and one in which the mathematical problems soon become extreme.

<u>4. Coherent Systems and Structures</u>. One of the acknowledged influences upon American Science during the nineteen-fifties was the sight of "Sputnik" flashing across the evening skies. The problems of building a perfectly operating complex system from components less than perfect was demonstrated on television for the public with the failures of the Vanguard rockets during the initial phases of our space program. When the nation announced its goal of setting a man on the moon within the next decade, the reliability problems became paramount for astronauts as well as contractors to the National Aeronautics and Space Administration.

As an illustration of the mutual stimulation that often occurs between theory and practice, the paper published in 1961 by Birnbaum, Esary and Saunders dealt with this problem and was entitled "Multi-component Systems and their Reliability", (see [6]). This paper first introduced the concept of coherent structures and their reliability when built from dichotomous state components with reliability less than unity. These results were an extension of an earlier paper by Moore and Shannon, (see [31]) treating a related problem for two-terminal networks.

The formulation was as follows: let the state of the ith component be X_i , a Bernoulli random variable counting 1 if good and 0 otherwise, then the state of the system is

$$\Phi = \phi(X_1, \ldots , X_n)$$

where ϕ is a structure function, i.e. a monotone Boolean function of n Boolean variates, and its reliability is

$$h(p_1, \ldots ,p_n) = E\phi(X_1, \ldots ,X_n)$$

where

$$EX_i = p_i \quad i = 1, \ldots , n$$

is the reliability of the ith component.

This simple formulation makes possible a clarification of the relationship between component reliability and system reliability as well as generalizations of the concepts of paths and cuts, length and width, of a structure, which were introduced earlier for two-terminal networks.

Fifteen years ago it was thought, since electrical connection could only be made in series or parallel, and therefore all two-terminal networks could be so constructed, that all systems of interest could be represented by series and parallel combinations of such networks. One of the main contributions of this paper was to point out that the concept of coherence (Esary's word) was the right concept to encompass most reliability problems and that there were actual engineering systems which could not be represented by series-parallel combinations.

The paper also defined combination of structures and the essentiality of components which allowed a representation of the structure in terms of its minimal paths or minimal cuts. Fault tree analysis is "merely" a systematic attempt to obtain the representation of a structure in terms of its minimal cuts and as such is, in Barlow's words, "the missing link between coherent structure theory and its practical utilization." This work continued with two papers: "On the Probabilistic theory of Complex Structures," see [7], and "Theory of Reliability for Coherent Structures," see [8], which systematically attacked the algebraic properties and the essentiality of components in conjunction with their probabilistic behavior.

The next paper, written with J. D. Esary, entitled "Modules of Coherent Binary Systems," see [9], introduced formally the concept of a module and clarified its connection with minimal paths. The utilization of the concept of coherence played a fundamental role in the determination of maximal modules for a system and its decomposition into disjoint modules. The intimate relationship of coherence with Boolean switching functions was also shown in this paper.

The structural properties of coherent systems are now often presented as the bases of the theory of reliability and life testing. In fact that topic now serves as the first chapter in the text *Statistical Theory of Reliability and Life Testing* by Barlow and Proschan, see [23].

5. Life Distributions for Coherent Structures. If $X_1(t)$ is a Bernoulli variate at each time t, which as before indicates by 1 or 0 whether or not the ith component is working or failed, we can consider as a function of time the system performance as a stochastic process

$$\Phi(t) = \phi(X_1(t), \ldots, X_n(t)) \qquad \text{for } t > 0$$

where ϕ is a coherent structure. The question arises: what are the probabilistic conditions on the component which will guarantee the existence of a system life? Intuitively, a device has a life if it functions continuously in time until failure occurs, after which it remains failed. Thus we say that a device with performance process $\{X(t), t \geq 0\}$ has a life if

$$P\{X(s) = 1 \text{ for all } s \text{ in } [0,t) | X(t) = 1\} = 1$$

for all t such that $P\{X(t) = 1\} > 0$. (The alternative to the non-existence of a system life is not eternal life but death and resurrection or E.S.P -- error some place).

If a component has a life then there exists a non-negative random variable T such that $X(t) = I\{ T > t \}$ is the indicator of the event "the life of the component exceeds t" and $P[X(t) = 1]$ is a proper survival distribution for $t \geq 0$. One proves easily that if all components have lives and ϕ is a coherent structure then the system Φ has a life. In [27] it was shown that, with weak additional assumptions, the converse is true and even the stronger result: if a system has a life then all of the components have lives and the structure is coherent.

This paper by Esary and Marshall was submitted to *Technometrics* but, I surmise because of the Editor's opinion of its limited interest and the increasing costs of publication, it was cut in half through the novel expedient of reducing the size of the type. The contents were actually difficult to read without a magnifying glass (see [27] if you own one). These authors are not the only ones who have been accorded the distinction of having their writ engraved on the head of a pin but I believe they were the first ones to be so treated in the first edition.

That was in 1964 and things were about to change. "Paul and Silas" were proclaiming the Gospel of Reliability. The "New Testament," namely *Mathematical Theory of Reliability* by Barlow and Proschan, see [22], was soon to be published. This basic text augmented the "Old Testament" called "Reliability Theory and Practice" by Bazovsky, see [24], which had been the first book emphasizing the special problems and techniques of Reliability *per se*.

During this time Professor Birnbaum published two papers entitled, respectively, "Some Concepts and Problems of a Mathematical Theory of Reliability," see [10], and "A Survey of Some Recent Results on Reliability of Structures," see [11], which both clarified and summarized the then current state of knowledge about the theory of reliability of structures. These papers also formulated some of the unsolved problems which had appeared by that time.

One of the questions which was discussed was the fundamental role which coherence seemed to play in system life. One particular question which arose was the following: what is the class of life lengths, to which if all components of a coherent structure belong, will insure that the system has a life length in the same class? This particular question was first asked by J. D. Esary in a discussion of a paper by

Barlow, pp 89-93, in a book published in 1963, see [36].

The complete answer was obtained by Birnbaum, Esary and Marshall in 1966, see [12]. It is the class of distributions with failure rate which increases on the average (IFRA) i.e. if q is the failure rate of the life of a component then the average failure rate $\bar{q}$ defined by

$$\bar{q}(t) = \frac{1}{t} \int_0^t q(x)\, dx$$

is an increasing function of $t > 0$.

This class has the following properties:

(i) This class contains the exponential class of distributions.

(ii) If ϕ is coherent and its components have independent lives each with an IFRA distribution then Φ has a life which has a distribution which is also IFRA.

(iii) It is the smallest class with properties (i) and (ii).

The class of IFRA variates introduced in this paper was thus intimately related to coherence and therefore important in its own right and, moreover, it contained as a subset the important IFR class which had been so extensively studied by Barlow, Proschan and others. See [22] and the references there.

One of the equivalent formulations of coherence, found in [6], was that

$$\phi \text{ is coherent iff } \quad \mathrm{COV}(\Phi,\ S - \Phi) > 0$$

where

$$\Phi = \phi(X_1, \ldots , X_m) \quad \text{and} \quad S = \sum_{i=1}^{m} X_i .$$

The idea of association of random variables as a generalization of independence, which has proved to be so fruitful in reliability applications, stemmed in part from this formulation of coherence. The relationship is clear since random variables $T_1, \ldots, T_n$ are *associated* whenever

$$\operatorname{cov}\left(\Gamma(\underline{T}), \Delta(\underline{T})\right) \geq 0$$

for all pairs of increasing binary functions Γ, Δ. (The restriction to be binary is no loss of generality.) The papers which first exploited this notion were by Esary, Proschan and Walkup, see [30], and Esary, Proschan, see [28].

Professor Birnbaum's next paper, published in the Fifth Berkeley Symposium, co-authored with J. D. Esary in [13], also exploited this equivalent formulation of coherence along with the additional knowledge of structure, such as its length and width, to obtain a set of bounds for the reliability function of any coherent system using a grid formed from coherent structures of identically distributed Bernoulli components.

Later Esary and Proschan in [29] applied the idea of association of component lives to improve and generalize the reliability bounds which Esary had obtained earlier with Birnbaum loc. cit.

6. The Importance of Components in a Structure. In two of his latest publications in this field Professor Birnbaum concentrated on the problem of measuring the importance of components and modules within a system. This determination is valuable if one hopes to increase the reliability of the system by increasing the reliability of certain of its components. One would hope to obtain the maximum improvement in system reliability for a given expenditure of money. These papers "On the Importance of Components in a System" see [15], and "On the Importance of Different Components in a Multi-component System," see [16], were the first steps in another important area of both practical and theoretical importance.

In these papers Birnbaum defines the *reliability importance* of the $\underline{i}$th component by

$$B(i|\underline{p}) = \frac{\partial h(\underline{p})}{\partial p_i}$$

where $\underline{p} = (p_1, \ldots, p_n)$ denotes the vector of component reliability for a coherent system and $h(\underline{p})$ the system reliability function. Clearly to evaluate $B(i|\underline{p})$ one must know $\underline{p}$. In the case where the vector $\underline{p}$ is not known, Birnbaum defines the *structural importance* of the $\underline{i}$th component by

$$B(i) = \left.\frac{\partial h(\underline{p})}{\partial p_i}\right|_{p_1 = \cdots = p_n = \frac{1}{2}} .$$

He then used this definition to obtain the importance of a module within a structure and gives applications of these results.

From this initial idea have sprung further developments. In a paper called "Reliability Growth" Proschan used this idea of importance to obtain the rate of improvement of the system reliability as the component is improved in reliability. (See [32]).

Benefiting from this attempt a new measure of component importance with certain improved properties was introduced subsequently by Barlow and Proschan in "Importance of System Component and Fault Tree Events," see [21].

7. Fatigue Life. In 1968 Dr. Birnbaum returned to a problem on which he had worked ten years previously, namely, the determination of the life of structural components subject to fatique.

In [14] it was shown by Birnbaum and Saunders that a rule accepted in engineering practice, called "Miner's Rule for Cumulative Fatigue Damage," which was supported by empirical evidence but without any but the most meager theoretical justification, could be derived as a close approximation to the expectation of the number of random addends necessary to reach a given sum, assuming they are IFR variates in a cyclic renewal process where elements with index differing by a given period are identically distributed. This work was continued in [33] by Saunders.

The next problem to be attacked was the derivation of the distribution of the number of such addends to reach that given sum under similar assumptions. This was accomplished in [17]. If the increments to crack growth per cycle are independent with mean μ and variance σ^2 then the random number N of such cycles necessary to reach a critical length ω for the first time has, by the central limit theorem, asymptotically the distribution

$$P[N \leq n] \cong \Phi\left[\frac{n\mu-\omega}{\sqrt{n}}\right] .$$

where Φ here denotes the standard normal distribution function.

The difference between the approach in [17] and earlier work, in which this distribution had been discovered, was the parameterization and interpretation. Let $\beta = \omega/\mu$, $\alpha = \sigma\sqrt{\mu\omega}$ then β is the median life and α is a shape parameter in the distribution.

$$\Phi\left[\frac{1}{\alpha}\xi(t/\beta)\right] \quad \text{for} \quad t > o.$$

where $\xi(x) = \sqrt{x} - (1/\sqrt{x})$ for $x > 0$.

This distribution, called by Dr. Birnbaum "new-fangled" was subsequently labeled by some persons "the Birnbaum-Saunders Distribution." (I know because I've overheard persons refer to it, just for convenience, by its initials. This is a common practice; I've alluded to joint work of Barlow and Scheuer using the same mnemonic device.)

This new-fangled distribution is related to the log-normal in that both transformations satisfy the functional equation $\xi(t) = -\xi(1/t)$ which is equivalent with a reciprocal relationship for the variates themselves. Moreover, in this form the estimation problems can be solved for the parameters α and β and simple, easily computed estimators, which are virtually equivalent with the maximum likelihood estimators, can be found, see [18].

Further work on this idea of reciprocal variates was accomplished in [34].

8. Conclusion. I have tried to sketch how Professor Birnbaum's work has been near the developing edge of Reliability theory during the past quarter century. It is difficult to reference all of the subsequent work which is partially derived from, or related to, that of Professor Birnbaum and I apologize to any author who has not been referenced and feels slighted that Birnbaum's ideas were not at least partially precedent to his own.

My introductory remarks half-wittedlv confusing mental with physical stature are transparent to anyone who ever conversed (in any of the Indo-European languages) with Professor Birnbaum on a mathematical subject, or on virtually any subject of cultural interest or social importance, for they would find him to be described, in an old phrase, "both a gentleman and a scholar" and a man of imposing stature.

Professor Birnbaum,* at this important milepost in your career, we share with you the honor of your accomplishments and the pride that must be yours at having been so instrumental in the genesis and development of Reliability Theory. On behalf of Professors Ron Pyke, Al Marshall, Richard Barlow, Ernie Scheuer, Janet Myhre and myself, all of whom were former students of yours, and on behalf of Professors James Esary and Frank Proschan, who are friends or co-authors or both, and other friends and students too numerous to mention here, I tender to you the respect and esteem which your years of outstanding work, in reliability theory in particular, and in science and engineering in general, now merits.

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*Professor Birnbaum was in the audience during this presentation.

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