Reading: BKRW Sections A.1, A.2, A.5; Chapter 3, Sections 1-2.
Supplementary Reading: VdVW, Chapter 3.11.
Due: 1/22/98

0. Tell me the title of your project and give a brief tentative outline. (Please hand this in next Tuesday, 1/20.)

1. A. Let P and Q be fixed measures with $\rho(P,Q) < 1$ (so that $H(P,Q) = 1 - \rho(P,Q) > 0$). Let P^n and Q^n be the probability distributions of $X_1, \cdots, X_n$ iid P or Q respectively. Show that $\rho(P^n, Q^n) = (\rho(P,Q))^n \to 0$ and hence that $H(P^n, Q^n) \to 1$. Note that 1 is the largest possible value of the Hellinger distance H (as defined *with* the factor of $1/2$); when it equals 1 the corresponding measures are concentrated on disjoint sets!
B. Suppose that $\mathbf{P} = \{P_\theta : \theta \in \Theta \subset R\}$ is a one-dimensional regular parametric model with common dominating measure μ; thus, with $\theta_n = \theta + hn^{-1/2}$,

$$
\begin{aligned}
&||\sqrt{n}(\sqrt{p(\cdot\,,\theta_n)} - \sqrt{p(\cdot\,,\theta)}) - \frac{1}{2}\,h\,\dot{\mathbf{l}}_\theta\,\sqrt{p(\cdot\,,\theta)}||_\mu^2 \\
&= \int |\sqrt{n}(\sqrt{p(\cdot\,,\theta_n)} - \sqrt{p(\cdot\,,\theta)}) - \frac{1}{2}\,h\,\dot{\mathbf{l}}_\theta\,\sqrt{p(\cdot\,,\theta)}|^2\,d\mu \\
&\to 0\,.
\end{aligned}
$$

Note that this assumption implies that

$$
\begin{aligned}
n\,H^2(P_{\theta_n}, P_\theta) &= \frac{n}{2}\int \{\sqrt{p(\theta_n)} - \sqrt{p(\theta)}\}^2\,d\mu \\
&\to \frac{h^2}{8}\int \dot{\mathbf{l}}_\theta^2\,p(\,\cdot\,,\theta)\,d\mu = \frac{h^2}{8}\,I(\theta)\,.
\end{aligned}
$$

2. Now let P_θ^n and $P_{\theta_n}^n$ denote the product measures corresponding to P_θ and P_{θ_n} (i.e. the distributions of $X_1, \cdots, X_n$ iid P_θ or P_{θ_n} respectively). Show that

$$
H^2(P_\theta^n, P_{\theta_n}^n) \to 1 - \exp(-\frac{h^2}{8}\,I(\theta)) \begin{cases} > 0 \\ < 1 \end{cases} \text{if } I(\theta) > 0\,,\ h > 0\,.
$$

Thus the product laws P_θ^n and $P_{\theta_n}^n$ corresponding to the "null hypothesis" P_θ and the "local alternatives" P_{θ_n} do not separate entirely for regular models.

3. Show that $H^2(P,Q)$ does not depend on the choice of the measure μ dominating P and Q.