Statistics 592C, Problem Set 2 Solutions
Wellner; 2/1/99

1. Problem 2.4.2 page 125, VdV and W: the $L_r(Q)-$ entropy numbers of the class $\mathcal{F}_M = \{f1\{F \le M\} : f \in \mathcal{F}\}$ are smaller than those for $\mathcal{F}$ for any probability measure Q and for numbers $M > 0$ and $r \ge 1$.

   Solution: Let $m = N(\epsilon, \mathcal{F}, L_r(Q))$. Then there exist $f_1, \ldots, f_m$ such that for any $f \in \mathcal{F}$ there is an f_i with

   $$Q(|f - f_i|^r) = \|f - f_i\|^r_{L_r(Q)} < \epsilon^r .$$

   But then

   $$Q(|f1\{F \le M\} - f_i1\{F \le M\}|^r) \le Q(|f - f_i|^r 1\{F \le M\}) \le Q(|f - f_i|^r) < \epsilon^r$$

   so that $f_1 1\{F \le M\}, \ldots, f_m 1\{F \le M\}$ yields an $\epsilon-$net for $\mathcal{F}_M$. Thus $N(\epsilon, \mathcal{F}_M, L_r(Q)) \le N(\epsilon, \mathcal{F}, L_r(Q))$.

2. Problem 2.4.3 page 125, VdV and W: If $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ are Glivenko-Cantelli classes of functions, then
   (i) $\{a_1 f_1 + a_2 f_2 : f_i \in \mathcal{F}_i, |a_i| \le 1\}$ is Glivenko-Cantelli.
   (ii) $\mathcal{F}_1 \cup \mathcal{F}_2$ is Glivenko-Cantelli.
   (iii) the class of all functions that are both the pointwise limit and the $L_1(P)$ limit of a sequence in $\mathcal{F}$ is Glivenko-Cantelli.

   Solution: (i) Let $\mathcal{G} \equiv \{a_1 f_1 + a_2 f_2 : f_i \in \mathcal{F}_i, |a_i| \le 1\}$. For a_1, a_2 with $|a_i| \le 1$, $i = 1, 2$, and $f_i \in \mathcal{F}_i$, $i = 1, 2$, we have

   $$\begin{aligned} |(\mathbb{P}_n - P)(a_1 f_1 + a_2 f_2)| &\le |a_1||(\mathbb{P}_n - P)(f_1)| + |a_2||(\mathbb{P}_n - P)(f_2)| \\ &\le |(\mathbb{P}_n - P)(f_1)| + |(\mathbb{P}_n - P)(f_2)| \end{aligned}$$

   and hence

   $$\begin{aligned} \|\mathbb{P}_n - P\|^*_{\mathcal{G}} &\le \|\mathbb{P}_n - P\|^*_{\mathcal{F}_1} + \|\mathbb{P}_n - P\|^*_{\mathcal{F}_2} \\ &\to_{a.s.} 0 + 0 . \end{aligned}$$

Thus $\mathcal{G}$ is Glivenko-Cantelli.

(ii) Since $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{G}$ as defined in (i), we have

$$\|\mathrm{P}_n - P\|^*_{\mathcal{F}_1 \cup \mathcal{F}_2} \leq \|\mathrm{P}_n - P\|^*_{\mathcal{G}} \to_{a.s.} 0$$

by (i) and hence $\mathcal{F}_1 \cup \mathcal{F}_2$ is Glivenko-Cantelli.

(iii) Let $cl(\mathcal{F})$ be the collection of all functions that are both pointwise and $L_1(P)$ limits of functions in $\mathcal{F}$. Thus if $g \in cl(\mathcal{F})$, then there exists a sequence $\{g_m\} \subset \mathcal{F}$ such that both $g_m(x) \to g(x)$ for all $x \in \mathcal{X}$ and $P|g_m - g| \to 0$. Consequently

$$|\mathrm{P}_n g_m - P g_m| \to |\mathrm{P}_n g - P g|$$

as $m \to \infty$, and hence

$$\sup_{m \geq 1} |\mathrm{P}_n g_m - P g_m| \geq |\mathrm{P}_n g - P g| \, .$$

Since $\{g_m\} \subset \mathcal{F}$, this implies that

$$\|\mathrm{P}_n - P\|_{\mathcal{F}} \geq |\mathrm{P}_n g - P g|$$

for any such g, and therefore

$$\|\mathrm{P}_n - P\|_{cl(\mathcal{F})} \leq \|\mathrm{P}_n - P\|_{\mathcal{F}} \, .$$

It therefore follows that

$$\|\mathrm{P}_n - P\|^*_{cl(\mathcal{F})} \leq \|\mathrm{P}_n - P\|^*_{\mathcal{F}} \to_{a.s.} 0$$

and hence $cl(\mathcal{F})$ is $P-$Glivenko-Cantelli.