Statistics 583, Problem Set 6

Wellner; 5/6/98

Reading: Chapter 7, sections 7.1- 7.4
Due: Wednesday, May 13, 1998

1. Consider the collection $\mathcal{F}_0$ of distribution functions F on R^+ with $0 < E_F X < \infty$ and $E_F X^2 < \infty$. Let $T(F) \equiv \sigma(F)/\mu(F)$ for $F \in \mathcal{F}_0$ where $\sigma^2(F) = Var_F(X)$ and $\mu(F) = E_F(X)$. This is the *coefficient of variation of* F. Find the influence function of $T(F)$.

2. Let $U_{m,n} \equiv T(\mathbb{F}_m, \mathbb{G}_n)$ where $T(F,G) = \int F dG = P(X \le Y)$ is the Mann-Whitney functional and $\mathbb{F}_m$ and $\mathbb{G}_n$ are the empirical df's of $X_1, \ldots, X_m$ i.i.d. with df F, $Y_1, \ldots, Y_n$ i.i.d. with df G.
A. Show that
$$mnU_{m,n} + n(n+1)/2 = W_{m,n} \equiv \sum_{j=1}^{n} Q_j = \sum_{j=1}^{n} R_{m+j}\,.$$
B. Show that $EU_{m,n} = P(X \le Y) = \int F dG$ and that
$$\begin{aligned} Var(\sqrt{mn}U_{m,n}) &= (n-1)\int (1-G)^2 dF + (m-1)\int F^2 dG \\ &\quad - (N-1)(\int F dG)^2 + \int F dG \\ &= (n-1)Var[1-G(X)] + (m-1)Var[F(Y)] \\ &\quad + \int F dG(1 - \int F dG)\,. \end{aligned}$$
C. When $F = G$ use the results of A and B to compute $E_{(F,F)} W_{m,n}$. (This should agree with calculations for the Wilcoxon rank sum form of the statistic under the null hypothesis via finite sampling calculations.)

3. **Optional bonus problem:** Suppose that $\mathcal{F}_0$ is the same class of distribution functions as in problem 1, but now consider the functional $T(F)$ defined for a fixed $x_0 \in R^+$ by
$$T(F) \equiv e_F(x_0) \equiv E_F(X - x_0 | X > x_0) = \frac{\int_{x_0}^{\infty}(1-F(t))dt}{1-F(x_0)}\,.$$
This functional is the *mean residual life* functional. Find the influence function of $T(F)$.