Statistics 583, Problem Set 2

Wellner; 4/8/98

Reading: Chapter 6, sections 6.1 and 6.2; Ferguson, MS, Chapter 5, sections 5.3, 5.4, and 5.5; Lehmann, TSH, Chapter 5, sections 11-13, pages 232 - 245.
Due: Wednesday, April 15, 1998

1. Problem 3, Ferguson, MS, page 233. What happens when $X_1, \ldots, X_m$ are i.i.d. Exponential(λ) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential(μ)?

2. For observations $\underline{X} = (X_1, \ldots, X_n)$, let $X_{(1)} \le \ldots \le X_{(n)}$ denote the *order statistics* of the X_i's ($X_{(i)} \equiv \mathbb{F}_n^{-1}(i/n)$, $i = 1, \ldots, n$) and let $\underline{R} = (R_1, \ldots, R_n)$ denote the *ranks*; defined by $X_i = X_{(R_i)}$, $i = 1, \ldots, n$ (if $X_i = X_j$ for some $i < j$, define the ranks by $R_i < R_j$ and $X_i = X_{(R_i)}$).
A. Suppose that $X_1, \ldots, X_n$ are i.i.d. $F \in \mathcal{F}_{ac}$ (the absolutely continuous df's F on R) with density f. Show that the order statistics $\underline{X}_{(\cdot)} \equiv (X_{(1)}, \ldots, X_{(n)})$ are independent of the ranks $\underline{R}$ and that the order statistics have joint density $\bar{p}$ given by
$$\bar{p}(\underline{x}_{(\cdot)}) = n! \prod_{i=1}^{n} f(x_{(i)}), \qquad -\infty < x_{(1)} < \ldots < x_{(n)} < \infty$$
while
$$P(\underline{R} = \underline{r}) = \frac{1}{n!}, \qquad \underline{r} \in \Pi \equiv \{ \text{ all permutations of } \{1, \ldots, n\}\} .$$
B. Show that A continues to hold for any joint distribution p of the $\underline{X}$ which is symmetric with respect to permutation of its coordinates: $p(\pi \underline{x}) = p(\underline{x})$ for all $\underline{x}$ and $\pi \in \Pi$ where $\pi x \equiv (x_{\pi(1)}, \ldots, x_{\pi(n)})$.
C. If the joint distribution p of $\underline{X}$ is general (not permutation symmetric), show that the joint density $\bar{p}$ of the order statistics is given by
$$\bar{p}(\underline{x}_{(\cdot)}) = \sum_{\pi \in \Pi} p(\pi \underline{x}_{(\cdot)}) ,$$
and
$$P(\underline{R} = \underline{r} | \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) = \frac{p(r \underline{x}_{(\cdot)})}{\bar{p}(\underline{x}_{(\cdot)})} .$$

3. Lehmann, TSH, problem 45, page 263. (The test is given in (54) on page 234; this corresponds to the test in (4), page 20, Chapter 6, my lecture notes.)

4. **Optional bonus problem:** Show that the UMPU tests you derived in problem 1 can be carried out unconditionally.