1. (Problem 11.4, page 150, Efron and Tibshirani). If $\theta(F) = \mu + \int \alpha dF$ and

$$\begin{aligned}\hat{\theta}_n &= \theta(IF_n) \\ &= \mu + \int \alpha\, dIF_n \\ &= \mu + \frac{1}{n}\sum_{i=1}^n \alpha(X_i) \;\equiv\; \mu + \overline{\alpha}_n\ ,\end{aligned}$$

then

$$Var_F(\hat{\theta}_n) \;=\; \frac{1}{n} Var_F(\alpha(X_1)) \;=\; \frac{1}{n}\int (\alpha(x) - \int \alpha dF)^2\, dF(x)\ .$$

[Note that it is the first equality in this line that is *not* available to us for many (in fact most!) estimators $\hat{\theta}_n$.] Hence the (ideal) bootstrap estimator of $se_F(\hat{\theta}_n) = \sqrt{Var_F(\hat{\theta}_n)}$ is

$$\begin{aligned}\hat{se}_{Boot} &= \sqrt{Var_{IF_n}(\hat{\theta}_n(IF_n^*))} \\ &= \sqrt{\frac{1}{n} Var_{IF_n}(\alpha(X_1^*))} \\ &= \left\{\frac{1}{n}\int (\alpha(x) \;-\; \int \alpha dIF_n)^2\, dIF_n(x)\right\}^{1/2} \\ &= \left\{\frac{1}{n^2}\sum_{i=1}^n (\alpha(X_i) \;-\; \overline{\alpha}_n)^2\right\}^{1/2}\ .\end{aligned}$$

The pseudo - values $\theta^*_{n,i}$ are given by

$$\theta^*_{n,i} \;=\; n\hat{\theta}_n \;-\; (n-1)\hat{\theta}_{n-1,i} \;=\; \alpha(X_i)$$

and hence the jackknife estimator of the standard error of $\hat{\theta}_n$ is

$$\begin{aligned}\hat{se}_{Jack} &= \left\{\frac{1}{n(n-1)}\sum_{i=1}^n (\theta^*_{n,i} \;-\; \overline{\theta}^*_n)^2\right\}^{1/2} \\ &= \left\{\frac{1}{n(n-1)}\sum_{i=1}^n (\alpha(X_i) \;-\; \overline{\alpha}_n)^2)^2\right\}^{1/2} \\ &= \left(\frac{n}{n-1}\right)^{1/2} \hat{se}_{Boot}\ .\end{aligned}$$

2. (Problem 11.10, page 151, Efron and Tibshirani). The quadratic statistic $\hat{\theta}_n$ in (11.18) is

$$\hat{\theta}_n = \mu + \frac{1}{n}\sum_{i=1}^{n}\alpha(X_i) + \frac{1}{n^2}\sum_{1 \le i < j \le n}\beta(X_i, X_j)$$

The mean of $\hat{\theta}_n$ is

$$\begin{aligned} E\hat{\theta}_n &= \mu + \int \alpha\, dF + \frac{n(n-1)/2}{n^2}\int\int \beta(x,y)\, dF(x)\, dF(y) \\ &\to \mu + \int \alpha\, dF + \frac{1}{2}\int\int \beta(x,y)\, dF(x)\, dF(y)\ . \end{aligned}$$

Although Efron and Tibshirani did not make this precise in their statement of the problem, I will take the latter to be $\theta(F)$ ; indeed this is what $\hat{\theta}_n$ is estimating since $\hat{\theta}_n \to_p \theta(F)$ . With this definition of $\theta(F)$ the bias of $\hat{\theta}_n$ is

$$\begin{aligned} b_n(F) &\equiv E_F\hat{\theta}_n - \theta(F) \\ &= \left(\frac{n(n-1)/2}{n^2} - \frac{1}{2}\right)\int\int \beta(x,y)dF(x)dF(y) \\ &= -\frac{1}{2n}\int\int \beta(x,y)dF(x)dF(y)\ . \end{aligned}$$

The (ideal) bootstrap estimator of $b_n(F)$ is

$$b_n(IF_n) = -\frac{1}{2n}\int\int \beta(x,y)dIF_n(x)dIF_n(y)\ .$$

To find the Jackknife estimator of $b_n(F)$ we compute the pseudo - values $\theta^*_{n,i}$ :

$$\begin{aligned} \theta^*_{n,i} &= n\hat{\theta}_n - (n-1)\hat{\theta}_{n-1,i} \\ &= \mu + \alpha(X_i) + \frac{1}{n}\sum_{i<j}\beta(X_i, X_j) - \frac{1}{n-1}\sum_{j<k,\, j,k \ne i}\beta(X_j, X_k) \\ &= \mu + \alpha(X_i) + \left(\frac{1}{n} - \frac{1}{n-1}\right)\sum_{j<k,\, j,\, k \ne i}\beta(X_j, X_k) \\ &\quad + \frac{1}{n}\left\{\sum_{k=i+1}^{n}\beta(X_i, X_k) + \sum_{j=1}^{i-1}\beta(X_j, X_i)\right\} \\ &= \mu + \alpha(X_i) + \frac{-1}{n(n-1)}\sum_{j<k}\beta(X_j, X_k) \\ &\quad + \left\{\frac{1}{n} - \left(\frac{1}{n} - \frac{1}{n-1}\right)\right\}\left\{\sum_{k=i+1}^{n}\beta(X_i, X_k) + \sum_{j=1}^{i-1}\beta(X_j, X_i)\right\} \\ &= \mu + \alpha(X_i) - \frac{1}{n(n-1)}\sum_{1 \le j < k \le n}\beta(X_j, X_k) \end{aligned}$$

$$+ \ \frac{1}{n(n-1)} \left\{ \sum_{k=i+1}^{n} \beta(X_i, X_k) \ + \ \sum_{j=1}^{i-1} \beta(X_j, X_i) \right\} .$$

Hence it follows that the Jackknife estimator $\overline{\theta}_n^{*}$ of $\theta(F)$ is given by

$$\begin{aligned}
\overline{\theta}_n^{*} \ &= \ \mu \ + \ \frac{1}{n} \sum_{i=1}^{n} \alpha(X_i) \ - \ \frac{1}{n(n-1)} \sum_{1 \le j < k \le n} \beta(X_j, X_k) \\
&\qquad + \ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{k=i+1}^{n} \beta(X_i, X_k) \ + \ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \beta(X_i, X_j) \\
&= \ \mu \ + \ \frac{1}{n} \sum_{i=1}^{n} \alpha(X_i) \ + \ \frac{1}{n(n-1)} \sum_{1 \le j < k \le n} \beta(X_j, X_k) \ .
\end{aligned}$$

[Note that $\overline{\theta}_n^{*}$ is *unbiased:* $E_F \overline{\theta}_n^{*} \ = \ \theta(F)$.] It follows that the Jackknife estimator of bias is

$$\begin{aligned}
\hat{b}_{Jack} \ &= \ \hat{\theta}_n \ - \ \overline{\theta}_n^{*} \\
&= \ \left\{ \frac{1}{n^2} \ - \ \frac{1}{n(n-1)} \right\} \sum_{1 \le j < k \le n} \beta(X_j \, , X_k) \\
&= \ - \ \frac{1}{n^2(n-1)} \sum_{1 \le j < k \le n} \beta(X_j \, , X_k) \\
&= \ - \ \frac{1}{2n^2(n-1)} \sum_{j \ne k} \beta(X_j, X_k) \qquad \text{if} \ \ \beta(x,y) \ = \ \beta(y,x) \\
&= \ - \ \frac{1}{2n^2(n-1)} \left\{ n^2 \int \int \beta(x,y) \, d\mathbb{F}_n(x) d\mathbb{F}_n(y) \ - \ n \int \beta(x,x) \, d\mathbb{F}_n(x) \right\} \\
&= \ - \ \frac{1}{2(n-1)} \left\{ \int \int \beta(x,y) \, d\mathbb{F}_n(x) d\mathbb{F}_n(y) \ - \ \frac{1}{n} \int \beta(x,x) \, d\mathbb{F}_n(x) \right\} .
\end{aligned}$$

Thus we have (under the assumption that $\beta$ is symmetric: $\beta(x,y) \ = \ \beta(y,x)$ ),

$$b_n(\mathbb{F}_n) \ - \ \frac{n-1}{n} \hat{b}_{Jack} \ = \ \frac{1}{2n^2} \int \beta(x,x) \, d\mathbb{F}_n(x) \ = \ O_p(n^{-2})$$

if $E|\beta(X_1 \, , X_1)| \ < \ \infty$. [Note that $\beta$ can be taken to be symmetric without loss of generality: if $\beta$ is not symmetric, then $\beta_s(x,y) \ \equiv \ \{\beta(x,y) + \beta(y,x)\}/2$ is symmetric and

$$\begin{aligned}
\int \int \beta_s(x,y) \, dF(x) \, dF(y) \ &= \ \frac{1}{2} \left\{ \int \int \beta(x,y) dF(x) dF(y) \ + \ \int \int \beta(y,x) \, dF(x) dF(y) \right\} \\
&= \ \int \int \beta(x,y) \, dF(x) \, dF(y) \ .]
\end{aligned}$$

3. A. If $T_n = n^{-1} \sum_{j=1}^{n} (X_j - \bar{X})^2$ , then

$$T^*_{n,i} = nT_n - (n-1)T_{n,i} = \frac{n}{n-1}(X_i - \bar{X})^2$$

and hence

$$\bar{T}^*_n = \frac{n}{n-1}\hat{\mu}_2 .$$

Furthermore,

$$\begin{aligned}
\hat{Var}_n &= \frac{1}{n(n-1)} \sum_{i=1}^{n} (T^*_{n,i} - \bar{T}^*_n)^2 \\
&= \frac{1}{n(n-1)} \sum_{i=1}^{n} T^{*2}_{n,i} - \frac{1}{n-1}(\bar{T}^*_n)^2 \\
&= \frac{1}{n(n-1)} \sum_{i=1}^{n} (\frac{n}{n-1}(X_i - \bar{X})^2)^2 - \frac{1}{n-1}(\frac{n}{n-1}\hat{\mu}_2)^2 \\
&= \frac{1}{n-1}\frac{n^2}{(n-1)^2}\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^4 - \frac{1}{n-1}\frac{n^2}{(n-1)^2}\hat{\mu}_2^2 \\
&= \frac{n^2}{(n-1)^3}(\hat{\mu}_4 - \hat{\mu}_2^2) .
\end{aligned}$$