1. A. (i) When $F(x) = 1 - e^{-\lambda x}$, then $F^{-1}(t) = (-1/\lambda)\log(1-t)$ and $E_F(X) = \int_0^1 F^{-1}(s)\,ds = (1/\lambda)$, so

$$L_t(F) = \int_0^t [-\log(1-s)]\,ds = t + (1-t)\log(1-t)\,.$$

A. (ii) When $F(x) = 1_{[c,\infty)}(x)$, then $F^{-1}(s) = c$, $0 < s \le 1$, so that the numerator of $L_t(F)$ is

$$\int_0^t F^{-1}(s)\,ds = ct\,,$$

while the denominator equals this with $t = 1$, namely c. Hence for any degenerate df F we have $L_t(F) = t$, $0 \le t \le 1$. [This is why L is used to measure inequity. A "perfectly equal" income distribution (concentrated at a point) gives the identity function as its Lorenz curve. Note that $L_F(t) \le t$ for all d.f.'s F: since $L_F'(t) = F^{-1}(t)/\mu$ is nondecreasing in t, $L_F(t)$ is a convex function of t. Since $L_F(1) = 1$ and $L_F(0) = 0$, it follows that $L_F(t) = L_F((1-t)0 + t1) \le (1-t)L_F(0) + tL_F(1) = t$.]

B. The functional $L_t(F)$ is *not* weakly continuous because the denominator,

$$\int_0^1 F^{-1}(s)\,ds = E_F X = \int_0^\infty x\,dF(x)$$

is not weakly continuous as was shown in Example 7.2.3..

C. The Gateaux derivative of $L_t(F)$ is given by

$$\dot{L}_t(F; G - F) = \frac{d}{du} L_t(F_u)|_{u=0}$$

where $F_u \equiv (1-u)F + uG$. Now

$$\begin{aligned}
\frac{d}{du}\int_0^t F_u^{-1}(s)\,ds|_{u=0} &= \int_0^t \frac{d}{du} F_u^{-1}(s)|_{u=0}\,ds \\
&= -\int_0^t \frac{(G-F)(F^{-1}(s))}{f(F^{-1}(s))}\,ds \\
&= \int_0^t [G(F^{-1}(s)) - s]\,dF^{-1}(s)
\end{aligned}$$

since

$$\frac{d}{du} F_u^{-1}(s)|_{u=0} = -\frac{(G-F)(F^{-1}(s))}{f(F^{-1}(s))}\,.$$

Hence

$$\frac{d}{du} L_t(F_u)|_{u=0} \;=\; \frac{\dfrac{d}{du}\displaystyle\int_0^t F_u^{-1}(s)\, ds|_{u=0}}{\displaystyle\int_0^1 F^{-1}(s)\, ds}$$

$$- \;\frac{\displaystyle\int_0^t F^{-1}(s)\, ds}{(\displaystyle\int_0^1 F^{-1}(s)\, ds)^2} \frac{d}{du}\int_0^1 F_u^{-1}(s)\, ds|_{u=0}$$

$$= \; -\frac{1}{\mu_F}\left\{\int_0^t [G(F^{-1}(s)) - s]\, dF^{-1}(s) \;-\; L_t(F)\int_0^1 [G(F^{-1}(s)) - s]\, dF^{-1}(s)\right\}$$

$$\equiv \; \dot{L}_t(F\,; G - F)\,.$$

Taking $G = \delta_x(y) = 1_{[x,\infty)}(y) = 1_{(-\infty,y]}(x)$ yields the influence function $\psi_F(x)$ of $L_t(F)$:

$$\psi_F(x) \;=\; -\frac{1}{\mu_F}(\int_0^t [1_{(-\infty,F^{-1}(s)]}(x) \;-\; s]\, dF^{-1}(s)$$

$$-\; L_t(F)\int_0^1 [1_{(-\infty,F^{-1}(s)]}(x) \;-\; s]\, dF^{-1}(s))$$

$$= \; -\frac{1}{\mu_F}\int_0^1 [1_{[0,t]}(s) \;-\; L_t(F)]\,[1_{(-\infty,F^{-1}(s)]}(x) \;-\; s]\, dF^{-1}(s)\,.$$

D. Since $F^{-1}(t)$ is Hadamard - differentiable (tangentially) for distributions F with $f(F^{-1}(t)) > 0$, it seems plausible that we might be able to prove asymptotic normality of $L_t(IF_n)$ based on Hadamard differentiability, or Hadamard differentiability tangentially to continuous functions, assuming $E_F X^2 < \infty$ and F^{-1} continuous on $(0,1)$. This would yield

$$(*) \qquad \sqrt{n}(L_t(IF_n) \;-\; L_t(F)) \;\to_d\; N(0\,, E_F\psi_F^2(X))$$

where ψ_F is the influence function calculated in C. The difficulty in this argument is the denominator of $L_t(IF_n)$, namely the mean functional. This term is *not* Hadamard differentiable wrt $||\cdot||_\infty$, but we know that the CLT works for the mean by a direct proof, so all together it seems quite plausible that (*) will hold. [In fact it can be proved that (*) holds, and even for the whole function $\{\sqrt{n}(L_t(IF_n) - L_t(F))\colon 0 \le t \le 1\}$, by using methods from empirical process theory.]

2. A. For $n = 2m$,

$$T_{n,i} \;=\; \begin{cases} X_{(m+1)}\,, & i \le m \\ X_{(m)}\,, & i > m \end{cases},$$

and $T_{n,\cdot} = (X_{(m)} + X_{(m+1)})/2$. Hence

$$n\,\hat{Var}_n \;=\; n-1\,\{m\,(X_{(m+1)} \;-\; \frac{1}{2}\,(X_{(m)} + X_{(m+1)}))^2$$

$$+\;\; m\,(X_{(m)} \;-\; \frac{1}{2}\,(X_{(m)} + X_{(m+1)}))^2\}$$

(1) $$= \; n(n-1)\{\; \frac{X_{(m+1)} \;-\; X_{(m)}}{2}\;\}^2 \,.$$

B. When $n = 2m$ and $T(F) \;=\; F^{-1}(1/2)$, we have $T(IF_n) \;=\; X_{(m)}$ and $T_{n,i}$ are exactly as in A above. Hence (1) continues to hold.
When $n = 2m-1$, then $T(IF_n) \;=\; X_{(m)}$,

$$T_{n,i} \;=\; \begin{cases} X_{(m)}\,, & i \le m-1 \\ X_{(m-1)}\,, & i \ge m \end{cases}\,,$$

and $T_{n,\cdot} \;=\; \{(m-1)X_{(m)} \;+\; mX_{(m-1)}\}/(2m-1)$. Therefore

$$n\,\hat{Var}_n \;=\; (n-1)\{(m\,-1)\{X_{(m)} - \frac{1}{2m-1}\,[(m-1)X_{(m)} + mX_{(m-1)}]\}^2$$

$$+\;\; m\{X_{(m-1)} - \frac{1}{2m-1}\,[(m-1)X_{(m)} + mX_{(m-1)}]\}^2\}$$

$$= \; \frac{(n-1)^2\,(n+1)}{n}\left\{\frac{X_{(m)} - X_{(m-1)}}{2}\right\}^2$$

$$\to_d \;\; \frac{1}{4f^2(F^{-1}(1/2))}\left(\frac{\chi_2^2}{2}\right)^2$$

just as before.
Remark: The only case left out in A and B is that of an odd sample size, $n = 2m-1$ in part A. In this case,

$$T_{n,i} \;=\; \begin{cases} (X_{(m)} + X_{(m+1)})/2\,, & i \le m-1 \\ (X_{(m-1)} + X_{(m+1)})/2\,, & i = m \\ (X_{(m-1)} + X_{(m)})/2\,, & i \ge m+1 \end{cases}\,.$$

Thus

$$T_{n,\cdot} \;=\; \frac{1}{n}\,\{\,\frac{(m-1)}{2}\,(X_{(m)} + X_{(m+1)})$$

$$+\; \frac{1}{2}\,(X_{(m-1)} + X_{(m+1)}) \;+\; \frac{(m-1)}{2}\,(X_{(m-1)} + X_{(m)})\}\,.$$

The analysis from this point proceeds not just by algebra, but by careful grouping of terms and observing which terms are negligible. I will not present a full analysis here, but will record the result:

$$n\,\hat{Var}_n \;=\; \frac{(m-1)m^2}{2n^3}\,\{n(X_{(m+1)} - X_{(m-1)})\}^2 \;+\; o_p(1)$$

$$\to_d \;\; \frac{1}{4\,f^2(F^{-1}(1/2))}\,(\frac{\chi_4^2}{4})^2$$

since, with $g \equiv F^{-1}$,

$$n(X_{(m+1)} - X_{(m-1)}) \rightarrow_d g'(1/2)\, W$$

where $W \stackrel{d}{=} Y_1 + Y_2 \sim Gamma(2,1)$ for independent exponential rv's Y_1, Y_2 , so that $2\,W \sim \chi_4^2$. Thus for this definition of the sample median, it is true that $n\, \hat{Var}_n = O_p(1)$ for the full sequence of nonnegative integers n but it converges in distribution to one limit as $n = 2m \rightarrow \infty$ and a different limit as $n = 2m - 1 \rightarrow \infty$.