1. To find the influence function of $T(F)$, let $F_t = (1-t)F + t\delta_x$. The distribution function corresponding to $G \equiv \delta_x$ is $1_{(-\infty, y]}(x)$, $y \in R$, so the left limit is $G_-(y) = 1_{[x < y]}$, and the corresponding "at risk" function $1 - G_-(y) = 1_{[x \ge y]} = 1_{[y,\infty)}(x)$. We need to compute

$$\begin{aligned}
\lim_{t \to 0} \frac{T(F_t) - T(F)}{t} &= \frac{d}{dt} T(F_t)|_{t=0} \equiv IC(x; T, F) \equiv \psi_F(x) \\
&= \frac{d}{dt}\left\{ \int_0^{t_0} \frac{1}{1-(F_t)_-} \, dF_t \right\}|_{t=0} \\
&= \int_0^{t_0} \frac{1}{1-F_-} \, d(\delta_x - F) + \int_0^{t_0} \frac{(\delta_{x-} - F_-)}{(1-F_-)^2} \, dF \\
&= \frac{1_{[0,t_0]}(x)}{1-F_-(x)} - \int_0^{t_0} \frac{1}{1-F_-} \, dF + \int_0^{t_0} \frac{1}{1-F_-} \, dF - \int_0^{t_0} \frac{(1-\delta_{x-})}{(1-F_-)^2} \, dF \\
&= \frac{1_{[0,t_0]}(x)}{1-F_-(x)} - \int_0^{t_0} \frac{1_{[x \ge y]}}{(1-F_-(y))^2} \, dF(y) \\
&= \begin{cases} \dfrac{1}{1-F_-(x)} - \displaystyle\int_0^{x} \frac{1}{(1-F_-)^2} \, dF, & \text{if } x \le t_0 \\ -\displaystyle\int_0^{t_0} \frac{1}{(1-F_-)^2} \, dF & \text{if } x > t_0 \end{cases} .
\end{aligned}$$

When F is continuous $F_- = F$ and the influence function computed above reduces to:

$$IC(x; T, F) = 1_{[x \le t_0]} - \frac{F(t_0)}{1-F(t_0)} 1_{[x > t_0]} .$$

Note that $E_F \psi_F(X) = 0$ and (in the case of a continuous d.f. F)

$$E_F \psi_F^2(X) = \frac{F(t_0)}{1-F(t_0)} .$$

2. A. $T(F,G) = \int F \, dG$ is weakly continuous at all pairs (F,G) with *no common discontinuity points.* Proof: suppose that $F_n \to_d F$ and $G_n \to_d G$ where F and G have no common discontinuity points. Then $F_n \times G_n \to_d F \times G$ on $R \times R$: i.e. with $X_n \sim F_n$ and $Y_n \sim G_n$ independent, $(X_n, Y_n) \to_d (X,Y) \sim F \times G$; here (X,Y) are independent with df's F and G respectively. Since F and G have no common discontinuities, the function $g(x,y) \equiv 1_{[x \le y]}$ is continuous a.e. $F \times G$: note that all the mass points of the distribution $F \times G$ on R^2 fall off the diagonal, so $P(X = Y) = \int \{F(x) - F(x-)\} \, dG(x) = 0$. Hence by the Helly-Bray theorem

(Proposition 2.3.7, chapter 2, page 13) it follows that

$$T(F_n, G_n) = E1_{[X_n \le Y_n]} = Eg(X_n, Y_n) \to Eg(X, Y) = T(F, G) .$$

Note that $T(F, G)$ can be discontinuous if F and G share a point of discontinuity. For example, if $X_n \sim \text{Uniform}(0, 1/n) \equiv F_n$ so that $X_n \to_d 0 \equiv X \sim \delta_0$, and let $Y_n \sim \text{Uniform}(-1/n, 0) \equiv G_n$ so that $Y_n \to_d 0 \equiv Y \sim \delta_0$. Then $T(F_n, G_n) = P(X_n \le Y_n) = 0$, but $T(F, G) = P(X \le Y) = 1$. Hence $T(F, G)$ is *not* weakly continuous at *all* (F, G).

On the other hand, $T(F, G)$ is continuous at *every* pair (F, G) with respect to the Kolmogorov distance: if $||F_n - F||_\infty \to 0$ and $||G_n - G||_\infty \to 0$, then

$$T(F_n, G_n) - T(F, G) = \int (F_n - F) dG_n + \int F\, d(G_n - G)$$

where

$$|\int (F_n - G_n)\, dG_n| \le ||F_n - G_n||_\infty \int dG_n = ||F_n - G_n||_\infty \to 0$$

and, using integration by parts (Proposition 1.4.1, chapter 1, page 17)

$$|\int F d(G_n - G)| = |-\int (G_n(x-) - G(x-))\, dF(x)| \le ||G_n - G||_\infty \to 0 .$$

B. One simple definition of the Gateaux derivative -- which does not account for different sample sizes, $m \ne n$, might be as follows: let $F_t \equiv (1-t)F + tF_1$ and $G_t \equiv (1-t)G + tG_1$ for df's F, F_1, G, G_1. Then

(a) $$\frac{d}{dt} T(F_t, G_t)|_{t=0} = \lim_{t \to 0} \frac{T(F_t, G_t) - T(F, G)}{t} \equiv \dot{T}(F, G, F_1 - F, G_1 - G) .$$

A definition which would account for different sample sizes m and n might be to take $t_m \equiv m^{-1/2}$, $t_n = n^{-1/2}$, and $t_N \equiv (N/mn)^{1/2}$ where $m/N \equiv \lambda_N \to \lambda \in (0, 1)$, and consider

(b) $$\lim_{m,n \to \infty} \frac{T(F_{t_m}, G_{t_n}) - T(F, G)}{t_N} \equiv \dot{T}(F, G; \alpha, \beta) .$$

We first suppose that $m = n$ and calculate (a): clearly

$$\begin{aligned} T(F_t, G_t) &= \int \{F + t(F_1 - F)\}\, d\{G + t(G_1 - G)\} \\ &= \int F\, dG + t \int (F_1 - F)\, dG + t \int F d(G_1 - G) \\ &\quad + t^2 \int (F_1 - F)\, d(G_1 - G) \end{aligned}$$

so

$$\begin{aligned}\frac{d}{dt}T(F_t\,,G_t)|_{t=0} &= \int (F_1 - F)dG \;+\; \int F\,d(G_1 - G)\\ &= \int (F_1 - F)dG \;-\; \int (G_1 - G)_- dF\\ &= \int G_- d(F_1 - F) \;+\; \int Fd(G_1 - G)\\ &\equiv \dot{T}(F, G; F_1 - F\,, G_1 - G)\,.\end{aligned}$$

When we evaluate the limit in (b) we obtain a weighted version of this, namely,

$$\sqrt{1-\lambda}\int (F_1 - F)dG \;+\; \sqrt{\lambda}\int F\,d(G_1 - G) \;\equiv\; \dot{T}(F, G; \alpha\,, \beta)$$

where $\alpha \;\equiv\; \sqrt{1-\lambda}(F_1 - F)$, $\beta \;\equiv\; \sqrt{\lambda}(G_1 - G)$,

C. If $F_1 = \delta_x$, $G_1 = \delta_y$, then, from (a)

$$\begin{aligned}\dot{T}(F, G; \delta_x - F\,, \delta_y - G) &\equiv IC(x, y; T, (F, G))\\ &= -\,(G_-(y) - \int G_- dF) \;+\; F(y) - \int F\,dG\,.\end{aligned}$$

Thus we would "guess" that the asymptotic variance of $T(I\!F_n\,, I\!G_n)$ -- note that we have taken $m = n$ here --

$$Var_{F,G}[IC(X, Y; T, (F, G))] \;=\; Var_F[G_-(X)] \;+\; Var_G[F(Y)]\,.$$

By the analogous calculation via (b), we might "guess" that the asymptotic variance of $T(I\!F_m\,, I\!G_n)$ would be, assuming that $\lambda_N \;\equiv\; m/N \;\rightarrow\; \lambda$,

$$Var_{F,G}[IC(X, Y; T, (F, G))] \;=\; (1-\lambda)Var_F[G_-(X)] \;+\; \lambda Var_G[F(Y)]\,.$$

Note that this agrees with the calculations of Corollary 7.4.12.