1. Here $EX_{ij} = \theta_i$, $j = 1, \cdots, n_i$, $i = 1, \cdots, I$. Note that

$$\sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - \theta_i)^2 = \sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - X_{i\cdot})^2 + \sum_{i=1}^{I} n_i(X_{i\cdot} - \theta_i)^2 .$$

Hence, it follows that the estimates of $\theta_i$ under the full model are

$$\hat{\theta}_i = X_{i\cdot} = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij} , \quad i = 1, \cdots, I$$

and

$$\min_{\theta}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - \theta_i)^2 = \sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - X_{i\cdot})^2 .$$

Under the hypothesis $H : \theta_1 = \cdots = \theta_I$

$$\sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - \theta)^2 = \sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - X_{\cdot\cdot})^2 + n(X_{\cdot\cdot} - \theta)^2 ;$$

hence

$$\begin{aligned}\hat{\hat{\theta}}_i \equiv \hat{\hat{\theta}} &= \sum_{i=1}^{I}\sum_{j=1}^{n_i} X_{ij} / \sum_{i=1}^{I} n_i = \sum_{i=1}^{I} n_i\hat{\theta}_i/n \\ &\equiv X_{\cdot\cdot}\end{aligned}$$

and

$$\min_{\theta \in H}\sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - \theta_i)^2 = \sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - \hat{\hat{\theta}})^2 .$$

Here $n = \sum_{i=1}^{I} n_i$, $k = I$, $r = I - 1$. Thus the $F$ statistic for testing the null hypothesis $H$ versus $K : \theta_i \neq \theta_j$ for some $i \neq j$ is

$$\begin{aligned} F &= \frac{\sum_{i=1}^{I}\sum_{j=1}^{n_i}(\hat{\theta}_i - \hat{\hat{\theta}}_i)^2/(I-1)}{\sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - X_{i\cdot})^2/(n-I)} \\ &= \frac{\sum_{i=1}^{I} n_i[X_{i\cdot} - \sum_{i=1}^{I} n_i\hat{\theta}_i/n]^2/(I-1)}{\sum_{i=1}^{I}\sum_{j=1}^{n_i}(X_{ij} - X_{i\cdot})^2/(n-I)}\end{aligned}$$

$$\sim \; F_{I-1\,,\,n-I}(\delta^2)$$

where

$$\delta^2 \;=\; \frac{\sum_{i=1}^{I} n_i[\theta_i - \sum_{i=1}^{I} n_i\theta_i/n]^2}{\sigma^2} \,.$$

2. A. Here $n = I + J$, $k = 4$, and $r = 1$. It is easily seen that

$$\hat{\alpha}_1 \;=\; \bar{X}\,, \qquad \hat{\beta}_1 \;=\; \overline{uX} \;\equiv\; \frac{1}{I}\sum_{i=1}^{I} u_i X_i\,;$$

and

$$\hat{\alpha}_2 \;=\; \bar{Y}\,, \qquad \hat{\beta}_2 \;=\; \overline{vY} \;\equiv\; \frac{1}{J}\sum_{j=1}^{J} v_j Y_j\,.$$

Under $H : \beta_1 = \beta_2$, we find that $\hat{\hat{\alpha}}_1 \;=\; \bar{X}$, $\hat{\hat{\alpha}}_2 \;=\; \bar{Y}$, and

$$\hat{\hat{\beta}}_1 \;=\; \hat{\hat{\beta}}_2 \;=\; \frac{\sum_{i=1}^{I} u_i X_i \;+\; \sum_{j=1}^{J} v_j Y_j}{I + J} \,.$$

Hence the $F-$statistic for testing $H$ versus $K : \beta_1 \neq \beta_2$ is

$$F \;=\; \frac{\sum_{i=1}^{I} (\hat{\alpha}_1 + \hat{\beta}_1 u_i \;-\; \hat{\hat{\alpha}}_1 - \hat{\hat{\beta}}_1 u_i)^2 \;+\; \sum_{j=1}^{J} (\hat{\alpha}_2 + \hat{\beta}_2 v_j \;-\; \hat{\hat{\alpha}}_2 - \hat{\hat{\beta}}_2 v_j)^2}{[\sum_{i=1}^{I} (X_i \;-\; \hat{\alpha}_1 + \hat{\beta}_1 u_i)^2 \;+\; \sum_{j=1}^{J} (Y_j \;-\; \hat{\alpha}_2 + \hat{\beta}_2 v_j)^2]/(I + J - 4)}$$

$$=\; \frac{\frac{IJ}{I+J}(\hat{\beta}_1 \;-\; \hat{\beta}_2)^2}{[\sum_{i=1}^{I} (X_i \;-\; \hat{\alpha}_1 + \hat{\beta}_1 u_i)^2 \;+\; \sum_{j=1}^{J} (Y_j \;-\; \hat{\alpha}_2 + \hat{\beta}_2 v_j)^2]/(I + J - 4)}$$

and the test becomes "reject $H$ if $F \;>\; F_{1,I+J-4,\alpha}$ ".

B. In this case $n = I + J$, $k = 4$, and $r = 2$. The estimators are the same under the "big model" as in part A. Under the hypothesis $H$ we find that

$$\hat{\hat{\alpha}}_1 \;=\; \hat{\hat{\alpha}}_1 \;=\; (I\bar{X} \;+\; J\bar{Y})/(I + J)\,.$$

and, as in A,

$$\hat{\hat{\beta}}_1 \;=\; \hat{\hat{\beta}}_2 \;=\; \frac{\sum_{i=1}^{I} u_i X_i \;+\; \sum_{j=1}^{J} v_j Y_j}{I + J} \,.$$

Thus the $F-$statistic becomes

$$F \;=\; \frac{\sum_{i=1}^{I} (\hat{\alpha}_1 + \hat{\beta}_1 u_i \;-\; \hat{\hat{\alpha}}_1 - \hat{\hat{\beta}}_1 u_i)^2 \;+\; \sum_{j=1}^{J} (\hat{\alpha}_2 + \hat{\beta}_2 v_j \;-\; \hat{\hat{\alpha}}_2 - \hat{\hat{\beta}}_2 v_j)^2}{[\sum_{i=1}^{I} (X_i \;-\; \hat{\alpha}_1 + \hat{\beta}_1 u_i)^2 \;+\; \sum_{j=1}^{J} (Y_j \;-\; \hat{\alpha}_2 + \hat{\beta}_2 v_j)^2]/(I + J - 4)}$$

$$= \frac{\frac{IJ}{I+J}\left\{(\hat\beta_1 - \hat\beta_2)^2 \ + \ (\hat\alpha_1 - \hat\alpha_2)^2\right\}/2}{[\sum_{i=1}^{I}(X_i \ - \ \hat\alpha_1 + \hat\beta_1 u_i)^2 \ + \ \sum_{j=1}^{J}(Y_j \ - \ \hat\alpha_2 + \hat\beta_2 v_j)^2]/(I + J - 4)} ,$$

and the test becomes "reject $H$ if $F > F_{2,I+J-4,\alpha}$ ".

3. In my solution I will reverse the roles of $F$ and $G$ from Ferguson ... so that $F <_s G$ . By Hoeffding's formula,

$$P_\theta(\underline{Q} = \underline{q}) \ = \ \frac{1}{\binom{N}{n}} E_{Uniform}\left\{\prod_{j=1}^{n} \psi'_\theta(U_{(q_j)})\right\}$$

where

$$\psi_\theta(u) \ = \ G_\theta \circ F^{-1}(u) \ = \ \frac{e^{\theta u} - 1}{e^\theta - 1}$$

for the first alternatives, and

$$\psi_\theta(u) \ = \ G_\theta \circ F^{-1}(u) \ = \ \frac{u}{[e^\theta(1-u)+u]}$$

in the case of the second alternatives. In either case, the locally most powerful rank test rejects for those values $\underline{q}$ of $\underline{Q}$ which make

$$\frac{\partial}{\partial\theta} P_\theta(\underline{Q} = q)|_{\theta = 0} \ = \ \frac{1}{\binom{N}{n}} E_{Uniform}\left\{\sum_{j=1}^{n} \frac{\partial}{\partial\theta}\psi'_\theta(U_{(q_j)})|_{\theta=0}\right\}$$

$$= \sum_{j=1}^{n} E_{Uniform}\left\{\frac{\partial}{\partial\theta}\psi'_\theta(U_{(q_j)})|_{\theta=0}\right\}$$

as large as possible. Hence it remains only to calculate

$$\phi(u) \ \equiv \ \frac{\partial}{\partial\theta}\psi'_\theta(u)|_{u=0}$$

and $E_{Uniform}\phi(U_{(i)})$ for the two alternatives in question.

(i) In the first case,

$$\psi'_\theta(u) \ = \ \frac{\theta\, e^{\theta u}}{e^\theta - 1} ,$$

and straightforward calculation yields

$$\frac{\partial}{\partial\theta}\psi'_\theta(u) \ = \ e^{\theta u}\frac{(e^\theta - 1)(1 + \theta u - \theta) - \theta}{(e^\theta - 1)^2} .$$

By applying L'Hopital's rule twice, we find that

$$\frac{\partial}{\partial\theta}\psi'_\theta(u)|_{u=0} \ = \ u \ - \ \frac{1}{2} .$$

Since $EU_{(i)} = i/(N+1)$ , the locally most powerful rank test of $H$ versus this alternative $K$ is the Wilcoxon test "reject $H$ if $S_N = \sum_1^n Q_j > k_\alpha$".

(ii) In the second case,

$$\psi'_\theta(u) = \frac{e^\theta}{[e^\theta(1-u)+u]^2} .$$

Hence

$$\frac{\partial}{\partial\theta} \psi'_\theta(u)|_{u=0} = 2u - 1 ,$$

and again the locally most powerful rank test is the Wilcoxon rank sum test.

As for interpretations of these alternatives, first note that the functions $\psi_\theta(u)$ are distribution functions on $[0,1]$ with densities $\psi'_\theta(u)$ .

(i) This alternative is the simplest exponential family density related to the uniform$(0,1)$ distribution: the density is of the form $p_\theta(u) \equiv \psi'_\theta(u) = c(\theta)\exp(\theta u)\, 1_{[0,1]}(u)$ where $c(\theta) = \theta/(e^\theta - 1)$ is the normalizing factor.

(ii) For this family, note that

$$1 - \psi_\theta(u) = \frac{e^\theta(1-u)}{e^\theta(1-u) + u} ,$$

and hence the *odds ratio* is

$$\frac{1-\psi_\theta(u)}{\psi_\theta(u)} = e^\theta \frac{1-u}{u} = e^\theta \cdot \text{the odds ratio for Uniform}(0,1) .$$

Thus this family is one with *proportional odds ratios.*

4. Here the finite population from which we are sampling can be taken to be just $\{1, \cdots, N\}$ . Hence we have

$$\bar{z}_N = \frac{1}{N}\sum_{i=1}^N i = \frac{N+1}{2} ,$$

$$\sigma_z^2 = \frac{1}{N}\sum_{i=1}^N i^2 - \bar{z}_N^2$$

$$= \frac{(N+1)(2N+1)}{6} - \frac{(N+1)^2}{4} = \frac{1}{12}(N^2-1) ;$$

and

$$\eta_N = \frac{\max_{i \le N} |i-(N+1)/2|^2}{N\sigma_z^2} = \frac{3(N-1)}{N(N+1)} = O(N^{-1}) \to 0 .$$

Hence by the WWNH CLT it follows that under $H : F = G$ we have, if

$0 < \underline{\lim}\, m/N \ \le\ \overline{\lim}\, m/N \ <\ 1$ ,

(**) $$\frac{\sqrt{n}(S_{N/n} \ -\ (N+1)/2)}{\sqrt{(1-\dfrac{n-1}{N-1})(N^2-1)/12}} \ \to_d \ N(0,1)\ .$$

One nice way to rewrite the left side is as

$$\frac{\sqrt{n}(S_{N/n} \ -\ (N+1)/2)}{\sqrt{(1-\dfrac{n-1}{N-1})(N^2-1)/12}} \ =\ \frac{S_N \ -\ n(N+1)/2}{\sqrt{mn(N+1)/12}}\ .$$

[Actually it is possible to relax the hypothesis that $0 < \underline{\lim}\, m/N \ \le\ \overline{\lim}\, m/N \ <\ 1$ ; the convergence in (**) holds as $m \bigwedge n \ \to\ \infty$ in any way under $F = G$ .]