1. A. Now $X\,,Y$ have joint density

$$p_{\lambda,\,\mu}(x,y) \;=\; \lambda\, e^{-\lambda x}\;\mu\, e^{-\mu y} 1_{[0,\infty)}(x)1_{[0,\infty)}(y)\ ,$$

so that if $c > 0$, $g_c(X,Y) = (cX\,,cY) \sim p_{\lambda/c\,,\,\mu/c}(x,y)$, and hence $\overline{g}(\lambda\,,\mu) = (\lambda/c\,,\mu/c)$. Note that $\delta(\lambda\,,\mu) \equiv \lambda/\mu = (\lambda/c)/(\mu/c) = \delta(\overline{g}(\lambda\,,\mu))$, so that the hypotheses $H : \delta \le 1$ and $K : \delta > 1$ are invariant. $T(X,Y) = X/Y$ and $\delta(\lambda,\mu) = \lambda/\mu$ are maximmal invariants on the sample space and parameter space respectively, and since $2\,\mu\,Y \sim \chi^2_2$ and $2\,\lambda\,X \sim \chi^2_2$ are independent,

$$T(X,Y) \;=\; \frac{2\lambda X}{2\mu Y}\,\frac{\mu}{\lambda} \;\sim\; \delta^{-1}\,F_{2,2}\ .$$

Since this family of distributions has monotone ($\downarrow$) likelihood ratio, it follows that the UMP G – invariant test of H versus K rejects H when $T < F_{2,2,\alpha}$ (where $P(F_{2,2} \le F_{2,2,\alpha}) = \alpha$, or when $T^{-1} = Y/X > F_{2,2,1-\alpha} = (1-\alpha)/\alpha$.

B. If $g(x,y) = (y,x)$ is also considered, then the transformation induced on the maximal invariant of A is given by

$$g(T) \;\equiv\; T(g(X,Y)) \;=\; T(Y,X) \;=\; \frac{Y}{X} \;=\; \frac{1}{T(X,Y)}\ .$$

Claim: $T \bigvee T^{-1}$ is the maximal invariant wrt this group.

Proof. $T \bigvee T^{-1}$ is invariant since $T^{-1} \bigvee T = T \bigvee T^{-1}$; and $T \bigvee T^{-1}$ is maximal since $T \bigvee T^{-1} = T^* \bigvee T^{*-1}$ implies that either $T = T^*$ or $T = T^{*-1}$.

The corresponding maximal invariant on the parameter space is $\delta \bigvee \delta^{-1} = \dfrac{\lambda}{\mu} \bigvee \dfrac{\mu}{\lambda} \equiv \nu$, and the hypotheses $H : \delta = 1$ and $K : \delta \ne 1$ are clearly invariant. When expressed in terms of ν the hypotheses become $H : \nu = 1$ versus $K : \nu > 1$. It remains to show that the maximal invariant has monotone likelihood ratio (MLR).

By direct calculation, for $t \ge 1$,

$$(*)\qquad 1 - F_\eta(t) \;=\; P_{\lambda,\,\mu}(\,T \bigvee T^{-1} > t\,) = \frac{2+\eta t}{1+\eta t+t^2} \qquad \text{with} \quad \eta \;\equiv\; \nu + \frac{1}{\nu}\ .$$

Hence $T \bigvee T^{-1}$ has density

$$f_\eta(t) \;=\; \frac{\eta(t^2+1)+4t}{(t^2+\eta t+1)^2}\,1_{[1,\infty)}(t)\ ,$$

so that for $\eta_1 < \eta_2$

$$\frac{f_{\eta_2}}{f_{\eta_1}}(t) \;=\; \{\frac{t^2+\eta_1 t+1}{t^2+\eta_2 t+1}\}^2 \,\frac{\eta_2(t^2+1)+4t}{\eta_1(t^2+1)+4t} \;\equiv\; g(t)^2\cdot h(t)$$

where

$$g'(t) \;=\; \frac{(\eta_2-\eta_1)(t^2-1)}{(t^2+\eta_2 t+1)^2} \;\geq\; 0 \quad \text{for} \quad t \;\geq\; 1$$

and

$$h'(t) \;=\; \frac{(\eta_2-\eta_1)(4\,t^2-1)}{(t^2+\eta_2 t+1)^2} \;\geq\; 0 \quad \text{for} \quad t \;\geq\; 1\,.$$

Hence the distribution of $T \bigvee T^{-1}$ has MLR and the UMP $G-$invariant test rejects H when $M \equiv T \bigvee T^{-1} > (2-\alpha)/\alpha$. Then, when $\nu = 1$, $\eta = 2$, and

$$P_{\eta\,=\,2}(\,M \;>\; \frac{2-\alpha}{\alpha}\,) \;=\; \alpha\,.$$

[Proof of (*): Since $T \sim \delta F_{2,2}$ where $P(F_{2,2} \leq x\,) = x/(1+x)$,

$$\begin{aligned} P(\,T \bigvee T^{-1} > t) &= P(T > t) \;+\; P(T < t^{-1}) \\ &= P(F_{2,2} > \frac{t}{\delta}) \;+\; P(F_{2,2} < \frac{1}{t\delta}) \\ &= \frac{1}{1+t/\delta} \;+\; \frac{(1/t\delta)}{1+(1/t\delta)} \\ &= \frac{\delta}{\delta+t} \;+\; \frac{1/\delta}{1/\delta+t} \\ &= \frac{2+(\delta+1/\delta)\,t}{1+(\delta+1/\delta)t+t^2}\,.\text{]} \end{aligned}$$

2. A. Let G be the permutation group

$$G \;=\; \{g : g(\underline{x}) \;=\; (x_{\pi(1)}\,,\cdots,x_{\pi(n)})\,,\;\; \pi \,\in\, \Pi\,\}\,.$$

Then, if $\underline{X} \sim P_\theta$, for $g \in G$, $g(\underline{X}) \sim P_{\overline{g}\theta}$ with $\overline{g}\theta = (\Delta\,, \nu\circ\pi = (\nu_{\pi(1)}\,,\cdots,\nu_{\pi(n)}))$. Thus the hypotheses are invariant under G. The order statistics are a $G-$MI. But of course the X_i's are *not* identically distributed. If $\underline{X}(\cdot) = (X_{(1)}\,,\cdots,X_{(n)})$ denotes the order statistics, then

$$\begin{aligned} f(\underline{x}_{(\cdot)}\,,\Delta,\nu\,) &= \sum_{\pi\,\in\,\Pi} f(\pi\underline{x}\,;\Delta\,,\nu) \\ &= \frac{1}{\Gamma(\alpha)^n\,\beta^{n\alpha}\exp(\,\alpha\,\Delta\sum b_{\nu_i})}\,(\prod_{i=1}^n x_{(i)})^{\alpha\,-\,1}\sum_{\pi\,\in\,\Pi}\exp\left\{-\frac{1}{\beta}\sum_{i=1}^n x_{(\pi(i))}\exp(\,-\,\Delta b_{\nu_i})\right\} \\ &= \frac{1}{\Gamma(\alpha)^n\,\beta^{n\alpha}\exp(\,\alpha\,\Delta\sum b_{\nu_i})}\,(\prod_{i=1}^n x_{(i)})^{\alpha\,-\,1}\sum_{\pi^*\,\in\,\Pi}\exp\left\{-\frac{1}{\beta}\sum_{i=1}^n x_{(i)}\exp(\,-\,\Delta b_{\pi^*(i)})\right\} \end{aligned}$$

with $\pi^* \equiv \pi^{-1}\circ\nu$. Note that this distribution depends only on the $\bar{G}-$MI Δ .

B. Now

$$\begin{aligned} l(\Delta\,|\,\underline{X}_{(\cdot)}) &= \log f(\underline{X}(\cdot)\,;\Delta) \\ &= \log\Big\{\sum_{\pi^*\in\Pi}\exp\Big\{-\frac{1}{\beta}\sum_{i=1}^{n}X_{(i)}\exp(-\Delta b_{\pi^*(i)})\Big\} \\ &\qquad -\alpha\Delta\sum_{i=1}^{n}b_i + \text{constant in } \Delta \end{aligned}$$

so that

$$\begin{aligned} \dot{\mathbf{l}}_{\Delta}(\underline{X}_{(\cdot)};\Delta=0) &= \frac{1}{\sum_{\pi^*}\exp[\cdots]|_{\Delta=0}} \\ &\quad\cdot\sum_{\pi^*\in\Pi}\exp[\cdots]|_{\Delta=0}\Big\{-\frac{1}{\beta}\sum_{i=1}^{n}X_{(i)}\,exp(-\Delta b_{\pi^*(i)})(-b_{\pi^*(i)})|_{\Delta=0}\Big\} \\ &\quad-\alpha\sum_{j=1}^{n}b_j \\ &= \frac{1}{\beta\,n!}\sum_{\pi^*}\{\sum_{i=1}^{n}X_{(i)}\,b_{\pi^*(i)}\} - \alpha\sum_{i=1}^{n}b_i \\ &= \frac{1}{\beta\,n!}\sum_{i=1}^{n}X_{(i)}\sum_{\pi^*}b_{\pi^*(i)} - \alpha\sum_{i=1}^{n}b_i \\ &= \frac{1}{\beta}\bar{b}\sum_{i=1}^{n}X_{(i)} - \alpha n\bar{b} = \frac{n\,\bar{b}}{\beta}(\bar{X} - \alpha\beta)\,, \end{aligned}$$

and hence the locally MP invariant test rejects for large values of $\sum_{i=1}^{n}X_i$.

3. Note that $0 \le Y_1 \le \cdots \le Y_N$ and

(1.2) $$Z_i = (N-i+1)(Y_i - Y_{i-1})\,, \quad i=1\,,\cdots,N$$

(with $Y_0 \equiv 0$). Let $g(\underline{Z}) \equiv \underline{Y}$ be the map defined by (1.1) so that $g^{-1}(\underline{Y}) = \underline{Z}$ is given in (1.2). Then the Jacobian of g^{-1} has entries $N\,,(N-1)\,,\cdots,1$ on the diagonal, entries $-(N-1)\,,\cdots,-2\,,-1$ below the diagonal, and zero elsewhere. Hence $\det(J_{g^{-1}}) = tr(J_{g^{-1}}) = N!$ and the density of $\underline{Y}$ is given by

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{Z}}(g^{-1}(\underline{y}))\det(J_{g^{-1}}) = N!\prod_{i=1}^{N}\exp(-(N-i+1)(y_i - y_{i-1}))$$

$$
\begin{aligned}
&= N!\exp(\,-\sum_{i=1}^{N}(N-i+1)(y_i-y_{i-1})) \\
&= N!\exp(\,-\sum_{i=1}^{N}(\sum_{j=i}^{N}1)\,(y_i-y_{i-1})) \\
&= N!\exp(\,-\sum_{j=1}^{N}\sum_{i\,\le\,j}(y_i-y_{i-1})) \\
&= N!\exp(\,-\sum_{j=1}^{N}y_j) \;=\; N!\,f(y_1)\cdots f(y_N)
\end{aligned}
$$

on the set $0 \le y_1 \le \cdots \le y_N < \infty$ where $f(x) = \exp(\,-x)1_{[0,\infty)}(x)$ is the standard exponential density. Hence $\underline{Y} \overset{d}{=} \underline{V}_{(\cdot)}$ where $V_1, \cdots, V_N$ are iid exponential(1).