1. A. If $X_1, \cdots, X_n$ are i.i.d. $N(\theta, \sigma_0)$, to find optimal tests ϕ we can reduce (by sufficiency) to consideration of $\overline{X} \sim N(\theta, \sigma_0^2/n)$. My favorite family of tests (in fact the most powerful tests) of H versus K are the tests $\phi_c(\underline{X}) = 1\{\overline{X} > c\}$. For these tests

$$\begin{aligned} R(0, \phi_c) &= E_0 \phi_c(\underline{X}) = P_0(\overline{X} > c) \\ &= P_0(\sqrt{n}(\overline{X} - 0)/\sigma_0 > \sqrt{n}c/\sigma_0) \\ &= 1 - \Phi(\sqrt{n}c/\sigma_0) \end{aligned}$$

and

$$\begin{aligned} R(1, \phi_c) &= E_1(1 - \phi_c(\underline{X})) \\ &= P_1(\overline{X} \le c) = P_1(\sqrt{n}(\overline{X} - 1) \le \sqrt{n}(c - 1)) \\ &= \Phi(\sqrt{n}(c-1)/\sigma_0) . \end{aligned}$$

Since these tests are MP for testing H versus K, there are no points with risks below the curve given by $\{(R(0, \phi_c), R(1, \phi_c)) : c \in R\}$; this is the lower boundary of the risk body. Note that the tests $\phi_{ignore}(\underline{X}) \equiv \alpha$ have risks $R(0, \phi_{ignore}) = \alpha$, $R(1, \phi_{ignore}) = 1 - \alpha$. Thus the line $\{(\alpha, 1-\alpha) : \alpha \in [0,1]\}$ is in the risk body. Furthermore, note that the tests $\phi_c{}'(\underline{X}) \equiv 1 - \phi_c(\underline{X}) = 1\{\overline{X} \le c\}$ are MP for testing $H : \theta = 0$ versus $K' : \theta = \theta_1 < 0$, and by the Karlin - Rubin theorem these tests minimize the power function at points $\theta = \theta_1$ in the class of all tests with fixed power function (say at α) at $\theta = \theta_0$. Since

$$Power_{\phi'_c}(\theta) = E_\theta \phi_c{}' = 1 - R(\theta, \phi_c) ,$$

this says that the tests $\phi_c{}'$ maximize $R(1, \phi_c)$ over tests ϕ with $R(0, \phi) = \alpha$. Hence there are no points in the risk body with risks above the curve given by $\{(1 - R(0, \phi_c), 1 - R(1, \phi_c)) : c \in R\}$.

B. As n grows or $\sigma_0 \to 0$ the risk body expands out toward the boundary of the square $[0,1]^2$; see the attached plots.

C. As $\theta_1 \to \theta_0 = 0$, the risk body contracts toward the diagonal line $(\alpha, 1-\alpha)$ -- since the testing problem becomes harder. See the plots on page
.

2. When X and Y are independent geometric random variables, the joint density (with respect to the product of counting measure on the non-negative integers) can be rewritten as

$$\begin{aligned} P_{\theta_1, \theta_2}(X = x, Y = y) &= (1-\theta_1)(1-\theta_2)\exp(x \log\theta_1 + y \log\theta_2) \\ &= (1-\theta_1)(1-\theta_2)\exp(x[\log\theta_1 - \log\theta_2] + (x+y)\log\theta_2) \end{aligned}$$

$$= (1-\theta_1)(1-\theta_2)\exp(U\theta + T\xi)$$

where $U(x,y) = x$, $\theta = \log(\theta_1/\theta_2)$, $T(x,y) = x+y$, and $\xi = \log\theta_2$.
(a) Testing $H: \theta_1 \le \theta_2$ versus $K: \theta_1 > \theta_2$ in this family is equivalent to testing

$$H: \theta \le 0,\ \xi = \text{anything} \quad \text{versus} \quad K: \theta > 0,\ \xi = \text{anything}.$$

On the boundary $\Theta_B = \{(\theta_1,\theta_2)\colon \theta_1 = \theta_2\}$, $T = X+Y$ is sufficient and complete. Hence by Theorem 2.4.2 the UMPU test of H versus K is given by

$$\phi(X,Y) = \begin{cases} 1 & \text{if } X > c(T) \\ \gamma(T) & \text{if } X = c(T) \\ 0 & \text{if } X < c(T) \end{cases}$$

where c and γ are determined by

(1) $$E_{\theta_0 = 0}\{\phi(X,Y)\,|\,T=t\} = \alpha = .2 .$$

Now by direct calculation

$$\begin{aligned} P(X = x\,|\,T=t) &= \frac{P_{\theta_1,\theta_2}(X=x\,,Y=t-x)}{P_{\theta_1,\theta_2}(T=t\,)} \\ &= \frac{\theta_1^x\theta_2^{t-x}}{\sum_{x'=0}^{t}\theta_1^{x'}\theta_2^{t-x'}} = \frac{(\theta_1/\theta_2)^x}{\sum_{x'=0}^{t}(\theta_1/\theta_2)^{x'}} \\ &= \frac{1}{t+1}\,, \quad x \in \{0,\cdots,t\} \quad \text{when} \quad \theta_1 = \theta_2 . \end{aligned}$$

Hence (1) becomes:

$$\frac{t - c(t)}{t+1} + \frac{\gamma(t)}{t+1} = .2 .$$

Thus we choose

$$c(t) = \inf\{k : t-k \le .2(t+1)\}\,, \qquad \gamma(t) = .2(t+1) - (t - c(t)) .$$

(b) Testing $H: \theta_1 = \theta_2$ versus $K: \theta_1 = \theta_2$ in this family is equivalent to testing

$$H: \theta = 0,\ \xi = \text{anything} \quad \text{versus} \quad K: \theta = 0,\ \xi = \text{anything}.$$

Hence by Theorem 2.4.2 the UMPU test of H versus K is given by

$$\phi(X,Y) = \begin{cases} 1 & \text{if } X > c_2(T) \text{ or } X < c_1(T) \\ \gamma_i(T) & \text{if } X = c_i(T)\,,\ i = 1,2 \\ 0 & \text{if } c_1(T) < X < c_2(T) \end{cases}$$

where c_i and γ_i are determined by

(2) $$E_{\theta_0 = 0}\{\phi(X,Y)\,|\,T=t\} = \alpha = .2 ,$$

and

(3) $$E_{\theta_0 = 0}\{X\phi(X,Y)\,|\,T=t\} = \alpha E_{\theta_0 = 0}(X\,|\,T=t) .$$

Since $(X\,|\,T=t) \sim$ Discrete Uniform on $\{0,\cdots,t\}$, (2) and (3) become

$$c_1(t) \;+\; t - c_2(t) \;+\; \gamma_1(t) + \gamma_2(t) \;=\; .2(t+1)\,,$$

and, abbreviating $c_i(t) \;\equiv\; c_i\,,\;\; \gamma_i(t) \;\equiv\; \gamma_i\,,$

$$(c_1-1)c_1 \;-\; c_2(c_2+1) \;+\; t(t+1) \;+\; 2c_1\gamma_1 \;+\; 2c_2\gamma_2 \;=\; \alpha t(t+1)\,.$$

(c) For functions of the form $g(\theta_1\,,\theta_2) \;=\; a\log\theta_1 + b\log\theta_2 - c$ for fixed numbers $a\,,b\,,c$ with at least one of $a\,,b$ different from 0 we can derive UMPU tests of $H:g(\theta_1\,,\theta_2) \;=\; 0$ versus $K:g(\theta_1\,,\theta_2)$ *not* $=0$. This can be seen as follows: $g(\theta_1\,,\theta_2) \;=\; 0$ is equivalent to $\log(\theta_1\theta_2^{b/a}) \;=\; c/a$ if a *not* $=0$. Then note that the exponential term of the joint density of $X\,,Y$ can be written as

$$\begin{aligned}
&\exp(x\log\theta_1 \;+\; y\log\theta_2)\\
&\quad=\; \exp(x\log(\theta_1\theta_2^{b/a}) \;+\; (y-(b/a)x)\log\theta_2)\\
&\quad\equiv\; \exp(\theta \;+\; T\xi)\,.
\end{aligned}$$

3. If $X_1\,,\cdots,X_m$ are i.i.d. $N(\mu\,,1)$ and $Y_1\,,\cdots,Y$ are i.i.d. $N(\eta\,,1)$, then the joint density is

$$\begin{aligned}
p_{\mu,\eta}(\underline{x}\,,\underline{y}) \;&=\; (2\pi)^{-(m+n)/2}\exp(-\frac{1}{2}\{\sum_{i=1}^{m}(x_i-\mu)^2 \;+\; \sum_{j=1}^{n}(y_j-\eta)^2\})\\
&=\; c(\mu,\eta)h(\underline{x}\,,\underline{y})\exp(\mu\sum_{1}^{m}x_i \;+\; \eta\sum_{j=1}^{n}y_j)\\
&=\; c(\mu,\eta)h(\underline{x}\,,\underline{y})\exp((\mu-\eta)\sum_{1}^{m}x_i \;+\; \eta(\sum_{i=1}^{n}x_i+\sum_{j=1}^{n}y_j))\\
&\equiv\; c(\mu,\eta)h(\underline{x}\,,\underline{y})\exp(\theta U \;+\; \xi T)
\end{aligned}$$

where $U \;\equiv\; \sum_1^m X_i\,,\;\; T \;\equiv\; \sum_1^m X_i + \sum_1^n Y_j\,,\;\; \theta\equiv\mu-\eta\,,\;\; \xi\equiv\;\eta\,.$ Now testing $H:\mu\le\eta$ versus $K:\mu>\eta$ is equivalent to testing

$$H:\theta\le 0\,,\;\; \xi = \text{anything} \qquad \text{versus} \qquad K:\theta>0\,,\;\; \xi = \text{anything}\,.$$

hence it follows from theorem 6.2.4.2 that the UMPU test of H versus K is given by

$$\phi(\underline{X}\,,\underline{Y}) \;=\; \begin{cases} 1 & \text{if } \;U \;>\; c(T)\\ \gamma(T) & \text{if } \;U \;=\; c(T)\\ 0 & \text{if } \;U \;<\; c(T)\end{cases}$$

where $c(T),\gamma(T)$ satisfy $E_0\{\phi(\underline{X}\,,\underline{Y})\,|\,T=t\} \;=\; \alpha\,.$ But in this problem it is quite easy to find the critical points unconditionally by use of Remark 6.2.4.3: letting $N \;\equiv\; m+n$

$$\begin{aligned} V \;\equiv\; h(U,T) &\equiv \sqrt{Nm/n}(U/m \;-\; T/N) \\ &= \sqrt{Nm/n}\{\bar{X}(1-\frac{m}{N}) \;-\; \frac{n}{N}\bar{Y}\} \\ &= \sqrt{\frac{mn}{N}}(\bar{X} \;-\; \bar{Y}) \end{aligned}$$

is independent of T (the complete and sufficient statistics for Θ_B) by Basu's theorem, h is increasing in U for fixed T, and $V \sim N(0,1)$. Hence the test

$$\phi(\underline{X}, \underline{Y}) \;=\; \begin{cases} 1 & \text{if } \sqrt{\frac{mn}{N}}(\bar{X}-\bar{Y}) \;>\; z_\alpha \\ 0 & \text{if } \sqrt{\frac{mn}{N}}(\bar{X}-\bar{Y}) \;\leq\; z_\alpha \end{cases}$$

is UMPU for testing H versus K.

4. I will prove C first; then A and B follow as corollaries:

C. Suppose that $\underline{X}$ has joint density p. Then for any set Borel set $A \subset \{\underline{x} \in R^n : x_1 < x_2 < \cdots < x_n\}$,

$$\begin{aligned} P(\underline{X}_{(\cdot)} \in A) &= \int_{[\underline{x}_{(\cdot)} \in A]} p(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\ &= \sum_{r \in \Pi} \int_{[R(\underline{x}) = r,\, \underline{x}_{(\cdot)} \in A]} p(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\ &= \sum_{r \in \Pi} \int_A p(x_{(r_1)}, \cdots, x_{(r_n)})\, dx_{(1)} \cdots dx_{(n)} \\ &= \int_A \bar{p}(x_{(1)}, \cdots, x_{(n)})\, dx_{(1)} \cdots dx_{(n)} \end{aligned}$$

where we have used the fact that the correspondence between $(x_1, \cdots, x_n)$ and $(x_{(1)}, \cdots, x_{(n)})$ is one-to-one and linear with Jacobian $=1$ on each subset $[R = r]$, $r \in \Pi$. This proves that

$$\bar{p}(\underline{x}_{(\cdot)}) \;=\; \sum_{\pi \in \Pi} p(\pi \underline{x}_{(\cdot)}) \,.$$

Similarly,

$$\begin{aligned} P(R = r, \underline{X}_{(\cdot)} \in A) &= \int_{[R = r,\, \underline{x}_{(\cdot)} \in A]} p(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\ &= \int_A p(x_{(r_1)}, \cdots, x_{(r_n)})\, dx_{(1)} \cdots dx_{(n)} \\ &= \int_A \frac{p(x_{(r_1)}, \cdots, x_{(r_n)})}{\bar{p}(x_{(1)}, \cdots, x_{(n)})}\, \bar{p}(x_{(1)}, \cdots, x_{(n)})\, dx_{(1)} \cdots dx_{(n)} \end{aligned}$$

since $\overline{p}(x_{(1)}, \cdots, x_{(n)}) = 0$ implies $p(x_{r(1)}, \cdots, x_{r(n)}) = 0$ for each $r \in R$. This implies that

$$P(\underline{R} = \underline{r} \mid \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) = \frac{p(r\underline{x}_{(\cdot)})}{\overline{p}(\underline{x}_{(\cdot)})} .$$

B. When $p(\underline{x}) = p(\pi\underline{x})$ for all $\pi \in \Pi$, then

$$\overline{p}(\underline{x}_{(\cdot)}) = n!p(\underline{x}_{(\cdot)}) ,$$

and

$$P(\underline{R} = \underline{r} \mid \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) = \frac{p(r\underline{x}_{(\cdot)})}{\overline{p}(\underline{x}_{(\cdot)})} = \frac{p(\underline{x}_{(\cdot)})}{n!p(\underline{x}_{(\cdot)})} = \frac{1}{n!} .$$

Hence R is independent of $\underline{X}_{(\cdot)}$, and $P(R = r) = 1/n!$ for each $r \in \Pi$.
A. This follows easily from B since, in this case,

$$p(\underline{x}) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} f(x_{\pi(i)}) = p(\pi\underline{x}) .$$