1. (i) A test ϕ is *unbiased* if $E_\theta \phi(X) \le \alpha$ for all $\theta \in \Theta_0$ and $E_\theta \phi(X) \ge \alpha$ for all $\theta \in \Theta_1$.
(ii) A test ϕ is *uniformly most powerful unbiased* (UMPU) if $E_\theta \phi \ge E_\theta \phi_0$ for all $\theta \in \Theta_1$ and for all other level α unbiased tests ϕ_0.
(iii) A test ϕ is *Similar On the Boundary* (SOB) if $E_\theta \phi(X) = \alpha$ for all $\theta \in \Theta_B \equiv \Theta_0 \cap \Theta_1$.
(iv) A function $T(x)$ is a maximal invariant with respect to a group G if it is invariant ($T(gx) = T(x)$ for all $g \in G$ and all $x \in \mathbf{X}$), and maximal ($T(x) = T(x^*)$ implies that $x^* = g(x)$ for some $g \in G$).
(v) A test ϕ is *invariant* ϕ with respect to a group G if $\phi(gx) = \phi(x)$ for all $g \in G$ and all $x \in \mathbf{X}$.

2. See Chapter 6 notes.

3. Suppose that $X_1, \cdots, X_m$ are i.i.d. exponential(λ), and that $Y_1, \cdots, Y_n$ are i.i.d. exponential(μ); thus the density of X_1 is $\lambda e^{-\lambda x} 1_{[0,\infty)}(x)$. Consider testing $H: \lambda \le \mu$ versus $K: \lambda > \mu$ [or, equivalently, $H: E(X) \ge E(Y)$ versus $K: E(X) < E(Y)$].

A. Show that this testing problem is invariant with respect to the group of scale changes, G given by $g_c(\underline{x}, \underline{y}) = (c\underline{x}, c\underline{y})$ where $c > 0$.

Solution: If $X \sim$ exponential(λ), then

$$\begin{aligned} P_\lambda(cX > t) &= P_\lambda(X > t/c) = \exp(-\lambda t/c) \\ &= \exp(-(\lambda/c)t) = P_{\lambda/c}(X > t), \end{aligned}$$

and similarly for $Y \sim$ exponential(μ). Hence the induced group on the parameter space is $\overline{g}(\lambda, \mu) = (\lambda/c, \mu/c)$. Note that for any $\overline{g} \in \overline{G}$ we have $\overline{g}\Theta_0 = \{(\lambda/c, \mu/c): \lambda \le \mu\} = \{(\lambda, \mu): \lambda \le \mu\} = \Theta_0$ and $\overline{g}\Theta = \{(\lambda/c, \mu/c): (\lambda, \mu) \in R^+ \times R^+\} = \Theta$. Hence the testing problem is invariant under the group G.

B. Find the UMP G-invariant test of H versus K. [Hint: You may use the fact that the family of distributions $\{\delta^{-1} F_{r,s}: \delta > 0\}$ has monotone likelihood ratio.]
[I should have said *monotone decreasing likelihood ratio.*]

Solution: By sufficiency we may reduce to consideration of $(S, T) \equiv (\sum_1^m X_i, \sum_1^n Y_j)$. The induced group G^* on the space of the sufficient statistic is given by $G^* = \{g^*(s,t) = (cs, ct): c > 0\}$, and the maximal invariant for the group G^* is $V \equiv S/T$; the corresponding $\overline{G}$ maximal invariant is $\delta = \lambda/\mu$. Now $2\lambda X_i \sim \chi_2^2$, and similarly $2\mu Y_j \sim \chi_2^2$. hence $2\lambda S \sim \chi_{2m}^2$ and $2\mu T \sim \chi_{2n}^2$. Hence

$$\frac{n}{m} V \; = \; \frac{\mu}{\lambda} \cdot \frac{(2\lambda S/2m)}{(2\mu T/2n)} \; = \; \delta^{-1} F_{2m,2n}$$

where $F_{2m,2n}$ has an $F-$ distribution with degrees of freeddom $2m\,, 2n$. Since the family $\delta^{-1} F_{r,s}$ has monotone decreasing monotone likelihood ratio, we conclude that the UMP G-invariant test of H versus K is given by "reject H if $nV/m \; < \; F_{2m,2n,\alpha}$ " where $P(F_{2m,2n} \; \le \; F_{2m,2n,\alpha}) \; = \; \alpha$. (Alternatively, "reject H if $m/(nV) \; = \; (n^{-1}T/m^{-1}S) \; > \; F_{2m,2n,1-\alpha}$ " where $P(F_{2m,2n} \; \ge \; F_{2m,2n,1-\alpha}) \; = \; \alpha$.

C. Specify as exactly as possible how you would carry out the test derived in B.

Solution: See part B above.

4. Suppose that $X_1\,, \cdots, X_m$ are i.i.d. exponential(λ), and that $Y_1\,, \cdots, Y_n$ are i.i.d. exponential(μ); thus the density of X_1 is $\lambda e^{-\lambda x} 1_{[0,\infty)}(x)$. Consider testing $H : \lambda \le \mu$ versus $K : \lambda > \mu$.

A. What is a sufficient statistic for Θ_B ?

Solution: $\Theta_B \; = \; \{(\lambda\,, \lambda) : \lambda > 0\}$, and it is clear that the sufficient statistic for the resulting collection of distributions is $S + T$ where $S = \sum X_i$, $T = \sum Y_j$.

B. Find the UMP unbiased test of H versus K -- explaining the steps -- and leaving the test in a conditional form. What is the conditional distribution needed to carry out the test?

Solution: First we rewrite the joint density of the data as follows:

$$\begin{aligned} p(\underline{x}\,, \underline{y}\,; \lambda\,, \mu) \; &= \; \lambda^m \mu^n \exp(\, - \, \lambda \sum x_i \; - \; \mu \sum y_j) \\ &= \; \lambda^m \mu^n \exp(\, - \, (\mu - \lambda) \sum y_j - \lambda (\sum x_i + \sum y_j)) \\ &= \; \lambda^m \mu^n \exp(\, \theta U \; + \; \xi V) \end{aligned}$$

where $\theta \; \equiv \; - \, (\mu - \lambda)$, $U \; \equiv \; \sum_1^n Y_j \; \equiv \; T$, $\xi \; = \; \lambda$, and $V \; \equiv \; S + T$. Since testing H versus K is equivalent to testing $\theta \; = \; (\lambda - \mu) \; \le \; 0$ versus $\theta > 0$, we see that the UMP unbiased test of H versus K is of the form

$$\phi(\underline{X}\,, \underline{Y}) \; = \; \begin{cases} 1 & \text{if } \; T \; > \; c_\alpha(V) \\ \gamma(V) & \text{if } \; T \; = \; c_\alpha(V) \\ 0 & \text{if } \; T \; < \; c_\alpha(V) \end{cases}$$

where $\gamma(V)$ and $c_\alpha(V)$ are chosen so that $E\{\phi(X, Y) \,|\, V\} \; = \; \alpha$. The conditional distribution of T given $V = S + T$ is that of the n-th order statistic in a sample of $m + n - 1$ Uniform$(0, V)$ random variables; equivalently

$$(*) \qquad T/V \; \overset{d}{=} \; U_{m+n-1:n} \, ,$$

the $n-$ th order statistic in a sample of $m + n - 1$ Uniform$(0, 1)$ random variables.

C. Can the UMP unbiased test you found in B be carried out unconditionally? If so, how?

Solution: One way is via (*); since the conditional distribution of T/V given V does not depend on V this is the unconditional distribution, namely $\mathrm{Beta}(n\,, m+n-1-n+1) = \mathrm{Beta}(n\,, m)$. Of course because of the fact that $T/V = (T/S)/(1+(T/S))$ is a monotone increasing function of T/S , and $(T/n)/(S/m)$ has an $F_{2n\,,2m}$ distribution under $\lambda = \mu$, this is the same as "reject if $(n^{-1}T/m^{-1}S) > F_{2n,2m,1-\alpha}$ ". Note that the UMP G-invariant test derived in problem 3 and the UMPU test derived in problem 4 are exactly the same.

5. Suppose that X_1 has continuous distribution function F and $Y_1\,, Y_2$ are independent of X_1 and themselves independent with distribution function $G = F^2$. Let $\underline{Q} = (Q_1\,, Q_2)$ denote the ordered Y ranks.

A. Is $G <_s F$?

Solution: No. $G(x) = F^2(x) \le F(x)$ with strict inequality if $F(x) \in (0,1)$. Hence $F <_s G$.

B. Compute the probabilities $P_{F,G}(\underline{Q} = \underline{q})$ for $\underline{q} \in \{(1,2),(1,3),(2,3)\}$.

Solution: Here $G = \psi(F)$ with $\psi(u) = u^2$. Hence $\psi'(u) = 2u$, and by Hoeffding's formula

$$P(\underline{Q} = \underline{q}) \;=\; \frac{1}{\binom{3}{2}}\, E \prod_{j=1}^{2} \psi'(U_{(q_j)}) \;=\; \frac{4}{3}\, E[\,U_{(q_1)} U_{(q_2)}] \,.$$

where $(U_{(1)}\,, U_{(2)}\,, U_{(3)})$ are the order statistics of a sample of 3 Uniform$(0,1)$ random variables. Thus we compute

$$\begin{aligned} E(U_{(1)}U_{(2)}) &= 3! \iiint_{0 \le u_1 \le u_2 \le u_3 \le 1} u_1 u_2 \, du_1 \, du_2 \, du_3 \\ &= 3! \, \frac{1}{2} \int_0^1 \int_0^{u_3} u_2^3 du_2 \, du_3 \;=\; \frac{3!}{8} \int_0^1 u_3^4 du_3 \;=\; \frac{3}{20} \,. \end{aligned}$$

Hence $P(\underline{Q} = (1,2)) \;=\; 4/20 \;=\; 1/5 \;=\; 3/15$. Similarly,

$$\begin{aligned} E(U_{(1)}U_{(3)}) &= 3! \iiint_{0 \le u_1 \le u_2 \le u_3 \le 1} u_1 u_3 du_1 du_2 du_3 \\ &= \frac{3!}{2} \int_0^1 \int_0^{u_3} u_2^2 du_2 \, u_3 du_3 \;=\; \int_0^1 u_3^4 du_3 \;=\; \frac{1}{5} \,. \end{aligned}$$

Hence $P(\underline{Q} = (1,3)) \;=\; 4/15$. Finally, for $\underline{q} = (2,3)$,

$$E(U_{(2)}U_{(3)}) \;=\; 3! \iiint_{0 \le u_1 \le u_2 \le u_3 \le 1} u_2 u_3 du_1 du_2 du_3$$

$$= 3! \int_0^1 \int_0^{u_3} u_2^2 du_2 \, u_3 du_3 \; = \; 2 \int_0^1 u_3^4 du_3 \; = \; \frac{2}{5} \, ,$$

and this yields $P(\underline{Q} = (2,3)) \; = \; 8/15$.

C. Use B to find the most powerful rank test of $F = G$ versus $G = F^2$ at level $\alpha = 1/3$.

Solution: Since $P(\underline{Q} = (2,3)) \; > \; P(\underline{Q} = (1,3)) \; > \; P(\underline{Q} = (1,2))$, we conclude that the most powerful rank test of $F = G$ versus $G = F^2$ at level $\alpha = 1/3$ is given by "reject H if $\underline{Q} = (2,3)$ ".

6. Suppose that $X \sim Binomial(m, p_1)$, $Y \sim Binomial(n, p_2)$ are independent. Consider testing $H : p_1 \leq p_2$ versus $K : p_1 > p_2$.

A. Show that the UMPU test ϕ of H versus K is of the form

$$\phi(X,Y) \; = \; \begin{cases} 1 & \text{if } \; Y \; > \; c_\alpha(T) \\ \gamma(T) & \text{if } \; Y \; = \; c_\alpha(T) \\ 0 & \text{if } \; Y \; < \; c_\alpha(T) \end{cases}$$

where $T \; = \; X + Y$ and $c_\alpha(T)$, $\gamma_\alpha(T)$ are determined by the conditional distribution of Y given T under the null hypothesis, namely, $(Y|T) \; \sim$ Hypergeometric($T \, , m + n \, , n$) ; i.e. the probability of drawing $Y = y$ white balls in T draws without replacement from an urn containing m black balls and n white balls.

Solution: It is clear that my statement of the solution is incorrect! Large values of Y relative to $T = X + Y$ would lead us to conclude $p_2 > p_1$ rather than the other way around. Hence we should be rejecting for large values of X ... or *small* values of Y. This falls out from the following solution in a direct way; note that I've organized it in terms of the "large values of X " approach, but it could be done equally well the other way. First we rewrite the joint density of X and Y as

$$P_{p_1, p_2}(X = x \, , Y = y) \; = \; \binom{m}{x}\binom{n}{y} p_1^x (1 - p_1)^{m-x} p_2^y (1 - p_2)^{n-y}$$

$$= \binom{m}{x}\binom{n}{y} q_1^m q_2^n \exp(x \log(p_1/q_1) \; + \; y \log(p_2/q_2))$$

$$= \binom{m}{x}\binom{n}{y} q_1^m q_2^n \exp(x\{ \log(p_1/q_1) - \log(p_2/q_2)\} \; + \; (x + y) \log(p_2/q_2))$$

$$= \binom{m}{x}\binom{n}{y} q_1^m q_2^n \exp(\theta U \; + \; \xi T)$$

where $U \; \equiv \; X$, $T \equiv X + Y$,

$$\theta \; \equiv \; \log \frac{(p_1/q_1)}{(p_2/q_2)} \, , \quad \text{and} \quad \xi \; \equiv \; \log(p_2/q_2) \, .$$

Now H versus K is equivalent to $\theta \leq 0$ versus $\theta > 0$ and $T = X + Y$ is sufficient and complete for $\Theta_B = \{(p_1, p_2) : p_1 = p_2\} = \{(p_1, p_2) : \theta(p_1, p_2) = 0\}$. Thus the UMPU test of H versus K is given by

$$\phi(X,Y) = \begin{cases} 1 & \text{if } X > c_\alpha(T) \\ \gamma(T) & \text{if } X = c_\alpha(T) \\ 0 & \text{if } X < c_\alpha(T) \end{cases}$$

where $c_\alpha(T)$ and $\gamma(T)$ are determined so that $E\{\phi(X,Y)\,|\,T\} = \alpha$ where $(X \mid T) \sim$ Hypergeometric$(T, m+n, m)$.

B. The "Fisher exact test" is the conservative version of the above test which takes $\gamma(T) = 1$ and chooses $c_\alpha(T)$ as small as possible to still have $E\{\phi(X,Y)|T\} \leq \alpha$. This turns out to be quite conservative. A less conservative approach is to use the finite-sampling normal theory approximation to obtain an approximate critical point. Explain how to use the W-W-N-H CLT to obtain this approximate critical point.

Solution: Here the finite population is $\{z_1, \cdots, z_N\} = \{1, \cdots, 1, 0, \cdots, 0\}$ where there are m balls labelled with a 1 and n balls labelled with a zero. Hence $\overline{z}_N = m/n$, and

$$\sigma_z^2 = \frac{1}{N}\{m(1 - m/N)^2 + n(0 - m/N)^2\} = \frac{mn}{N^2}.$$

Since the Noether condition is satisfied trivially in this case, $\overline{Y} = X/T$ satisfies

$$\frac{X/T - m/N}{\sqrt{(1 - \frac{T-1}{N-1})\frac{mn/N^2}{T}}} \to_d N(0,1),$$

and this leads to choosing the approximate critical point

$$(\#) \qquad \tilde{c}_\alpha(T) = T\left\{\frac{m}{N} + z_\alpha\sqrt{(1 - \frac{T-1}{N-1})\frac{mn/N^2}{T}}\right\};$$

now the test becomes "reject H if $X > \tilde{c}_\alpha(T)$ " with $\tilde{c}_\alpha(T)$ given by (#).