Due: Wednesday, May 7, 1997

Reading: Lecture Notes, Chapter 7.
Huber, *Robust Statistics*, Chapters 1 and 2;
Efron and Tibshirani, *Bootstrap*, Chapter 21, sections 1-3.

1. Let $U_{m,n} \equiv T(\mathbb{F}_m, \mathbb{G}_n)$ where $T(F,G) = \int F\,dG = P(X \le Y)$ is the Mann-Whitney functional and $\mathbb{F}_m$ and $\mathbb{G}_n$ are the empirical df's of $X_1, \cdots, X_m$ i.i.d. with df F, $Y_1, \cdots, Y_n$ i.i.d. with df G.
A. Show that

$$mnU_{m,n} + n(n+1)/2 = W_{m,n} \equiv \sum_{j=1}^{n} Q_j = \sum_{j=1}^{n} R_{m+j}\,.$$

B. Show that $EU_{m,n} = P(X \le Y) = \int F\,dG$ and that

$$\begin{aligned} Var(\sqrt{mn}\,U_{m,n}) &= (n-1)\int (1-G)^2\,dF + (m-1)\int F^2\,dG \\ &\quad - (N-1)(\int F\,dG)^2 + \int F\,dG \\ &= (n-1)\,Var[1-G(X)] + (m-1)\,Var[F(Y)] \\ &\quad + \int F\,dG\,(1 - \int F\,dG)\,. \end{aligned}$$

C. When $F = G$ use the results of A and B to compute $E_{(F,F)}W_{m,n}$. (This should agree with the calculations you carried out in problem 4.4!)

2. Let F be a distribution function on R^2 with finite second moments, and let $\gamma_1(F)$, $\gamma_2(F)$ be the skewness and kurtosis of F:

$$\gamma_1(F) = \frac{E_F(X - \mu_F)^3}{\{Var_F(X)\}^{3/2}}\,,$$

$$\gamma_2(F) = \frac{E_F(X - \mu_F)^4}{\{Var_F(X)\}^{2}} - 3\,.$$

A. Find a collections of distribution functions $\mathbf{F}_1$ and $\mathbf{F}_2$ so that γ_1 and γ_2 are weakly continuous on $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively.
B. Invent a measure of skewness, $\tilde{\gamma}_1(F)$ which is weakly continuous on the set of all distribution functions.

3. For distribution functions F on R^+ and $t_0 > 0$, consider the functional $T(F)$ defined by

$$T(F) \;=\; \Lambda(t_0) \;=\; \int_0^{t_0} \frac{1}{1 - F_-}\, dF \;.$$

Find the influence function of $T(F)$.

4. Extra credit bonus problem: Consider point masses $P \equiv \delta_x$ and $Q \equiv \delta_y$ where $x\,, y \in S$ and $(S\,, d)$ is a metric space. Show that as $d(x,y) \to 0$, $d_{Pr}(P,Q)/d_{BL}(P,Q) \;\to\; 1$. But for $\mu \equiv (P + Q)/2$, $\mu_n \equiv \mu + (P - Q)/n$, and $d(x,y) = 1/n$, show that $d_{BL}(\mu_n\,, \mu)/d_{Pr}(\mu_n\,, \mu)^2 \;\to\; 1$ as $n \to \infty$.