Due: Wednesday, April 16, 1997

Reading: Lecture Notes, Chapter 6, section 6.3.
Ferguson, Mathematical Statistics, chapter 4, section 4.1, pages 143 - 148, chapter 5, section 5.6, pages 243 - 248.
[Lehmann, TSH, Chapter 6.]

1. A. Suppose that $Z_1, \cdots, Z_N$ are i.i.d. Pareto(θ) with density $p_\theta(x) = \theta x^{-(\theta+1)} 1_{[1,\infty)}(x)$, $\theta > 0$, and consider the joint density $p_\theta(\underline{z})$. Is $\{p_\theta : \theta \in (0, \infty)\}$ an exponential family? If so, identify the various components in the exponential family form explicitly.
   B. Now suppose that we condition on $(Z_{(1)}, \cdots, Z_{(N)}) = \underline{z}$, the order statistics of the Z_i's in A, and we sample $n < N$ of the z_i's without replacement; call the resulting sample $Y_1, \cdots, Y_n$.
   (i) What are the (conditional) mean μ_N and variance σ_N^2 of $\overline{Y}_n = n^{-1} \sum_{i=1}^n Y_i$?
   (ii) Under what further conditions does $(\overline{Y}_n - \overline{z}_N)/\sigma_N \rightarrow_d N(0,1)$ in probability under the model in A? Explain what this means and, in particular, specify for what values of θ the asymptotic normality holds and for what values of θ the asymptotic normality fails.
   C. Now suppose that $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ are i.i.d. $F \in \mathbf{F}_c$ under H_c while under K_1 the X_i's are i.i.d. Pareto(θ_1) and the Y_j's are i.i.d. Pareto(θ_2) with $\theta_1 > \theta_2$. Find a most powerful similar test of H_c versus K_1.

2. Suppose that $X_1, \cdots, X_n$ are i.i.d. with density
$$p_\theta(x) = p(x;\theta) = f(x - \theta)$$
   where
$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2} \, .]$$
   Note that $F(x) = 1/(1 + e^{-x})$ and hence that $f(x) = F(x)(1 - F(x))$.
   A. Use the generalized NP lemma to find the locally most powerful test of $H : \theta = 0$ versus $K : \theta > 0$ and show how to approximate the appropriate critical points to carry out your test.
   B. If instead of the null hypothesis $H : \theta = 0$ in A, we consider testing the much larger null hypothesis $H_s : X_1, \cdots, X_n$ are i.i.d. $F \in \mathbf{F}_s$, the collection of all continuous distribution functions symmetric about 0 and seek a locally most powerful test against the logistic alternatives, how would you

proceed to construct a LMP level α similar test?
[What is a sufficient statistic for $\mathbf{F}_s$?]