Due: Wednesday, April 9, 1997

Reading: Lecture Notes, Chapter 6, section 6.2; Lecture Notes, Chapter 13, section 13.5.
Ferguson, Mathematical Statistics, chapter 5, pages 215 - 242.
[Lehman, TSH, Chapter 4, pages 134 - 187.]

1. Suppose that $X_1, \cdots, X_n$ are i.i.d. $N(\theta, \sigma^2)$.
   A. Suppose that $\sigma = \sigma_0$ is known. Consider testing $H : \theta = \theta_0 = 0$ versus $K : \theta = \theta_1 = 1$. In the spirit of chapter 5, plot $(R(\theta_0, \phi), R(\theta_1, \phi))$ for your favorite family of tests ϕ. Find the entire risk body and plot it.
   B. What happens to the risk body as n grows or as $\sigma_0 \to 0$?
   C. What happens to the risk body as θ_1 decreases toward $\theta_0 = 0$?
2. Problem 5, Ferguson, MS, page 234.
3. Problem 6, Ferguson, MS, page 234.
4. For observations $\underline{X} = (X_1, \cdots, X_n)$, let $X_{(1)} \le \cdots \le X_{(n)}$ denote the *order statistics* of the X_i's ($X_{(i)} \equiv I\!F_n^{-1}(i/n)$, $i = 1, \cdots, n$) and let $\underline{R} = (R_1, \cdots, R_n)$ denote the *ranks*; defined by $X_i = X_{(R_i)}$, $i = 1, \cdots, n$ (if $X_i = X_j$ for some $i < j$, define the ranks by $R_i < R_j$ and $X_i = X_{(R_i)}$).
   A. Suppose that $X_1, \cdots, X_n$ are iid $F \in \mathbf{F}_{ac}$ (the absolutely continuous df's F on R) with density f. Show that the order statistics $\underline{X}_{(\cdot)} = (X_{(1)}, \cdots, X_{(n)})$ are independent of the ranks $\underline{R}$ and that the order statistics have joint density $\overline{p}$ given by

$$\overline{p}(\underline{x}_{(\cdot)}) = n! \prod_{i=1}^{n} f(x_{(i)}), \qquad -\infty < x_{(1)} < \cdots < x_{(n)} < \infty$$

   while

$$P(\underline{R} = \underline{r}) = \frac{1}{n!}, \quad \underline{r} \in \Pi \equiv \{\text{all permutations of } \{1, \cdots, n\}\}.$$

   B. Show that A continues to hold for any joint distribution p of the $\underline{X}$ which is symmetric with respect to permutation of its coordinates: $p(\pi \underline{x}) = p(\underline{x})$ for all $\underline{x}$ and $\pi \in \Pi$ where $\pi x \equiv (x_{\pi(1)}, \cdots, x_{\pi(n)})$.
   C. If the joint distribution p of $\underline{X}$ is general (not permutation symmetric), show that the joint density $\overline{p}$ of the order statistics is given by

$$\overline{p}(\underline{x}_{(\cdot)}) = \sum_{\pi \in \Pi} p(\pi \underline{x}_{(\cdot)}),$$

   and

$$P(\underline{R} \;=\; \underline{r} \mid \underline{X}_{(\cdot)} = \underline{x}_{(\cdot)}) \;=\; \frac{p(r\,\underline{x}_{(\cdot)})}{\overline{p}(\underline{x}_{(\cdot)})}\ .$$