1. (25 points) **Define** the following terms. In each case, provide an appropriate context for your definition.
 (i) An *unbiased test* ϕ.
 (ii) A *uniformly most powerful unbiased (UMPU) test* ϕ.
 (iii) A *Similar On the Boundary* (SOB) test ϕ.
 (iv) A maximal invariant with respect to a group G.
 (v) An *invariant test* ϕ with respect to a group G.

2. (24 points) **State two of the following four theorems**. In each case, provide an appropriate context for your statement.
 (i) The Wald-Wolfowitz-Noether-Hajek finite sampling central limit theorem.
 (ii) A theorem concerning existence of a UMP G-invariant test ϕ.
 (iii) A theorem concerning existence of a UMP unbiased test ϕ.
 (iv) Hoeffding's formula for the distribution of ranks (under the alternative).

Do **any two** of problems 3 - 6.

3. (26 points) Suppose that $X_1, \cdots, X_m$ are i.i.d. exponential(λ), and that $Y_1, \cdots, Y_n$ are i.i.d. exponential(μ); thus the density of X_1 is $\lambda e^{-\lambda x} 1_{[0,\infty)}(x)$. Consider testing $H: \lambda \le \mu$ versus $K: \lambda > \mu$.
 A. Show that this testing problem is invariant with respect to the group of scale changes, G given by $g_c(\underline{x}, \underline{y}) = (c\underline{x}, c\underline{y})$ where $c > 0$.
 B. Find the UMP G-invariant test of H versus K. [Hint: You may use the fact that the family of distributions $\{\delta^{-1} F_{r,s} : \delta > 0\}$ has monotone likelihood ratio.]
 C. Specify as exactly as possible how you would carry out the test derived in B.

4. (26 points) Suppose that $X_1, \cdots, X_m$ are i.i.d. exponential(λ), and that $Y_1, \cdots, Y_n$ are i.i.d. exponential(μ); thus the density of X_1 is $\lambda e^{-\lambda x} 1_{[0,\infty)}(x)$. Consider testing $H: \lambda \le \mu$ versus $K: \lambda > \mu$.
 A. What is a sufficient statistic for Θ_B?
 B. Find the UMP unbiased test of H versus K -- explaining the steps -- and leaving the test in a conditional form. What is the conditional distribution needed to carry out the test?
 C. Can the UMP unbiased test you found in B be carried out unconditionally? If so, how?

5. (26 points) Suppose that X_1 has continuous distribution function F and Y_1, Y_2 are independent of X_1 and themselves independent with distribution function $G = F^2$. Let $\underline{Q} = (Q_1, Q_2)$ denote the ordered Y ranks.
A. Is $G <_s F$?
B. Compute the probabilities $P_{F,G}(\underline{Q} = \underline{q})$ for $\underline{q} \in \{(1,2),(1,3),(2,3)\}$.
C. Use B to find the most powerful rank test of $F = G$ versus $G = F^2$ at level $\alpha = 1/3$.

6. (26 points) Suppose that $X \sim Binomial(m, p_1)$, $Y \sim Binomial(n, p_2)$ are independent. Consider testing $H : p_1 \leq p_2$ versus $K : p_1 > p_2$.
A. Show that the UMPU test ϕ of H versus K is of the form

$$\phi(X, Y) = \begin{cases} 1 & \text{if } Y > c_\alpha(T) \\ \gamma(T) & \text{if } Y = c_\alpha(T) \\ 0 & \text{if } Y < c_\alpha(T) \end{cases}$$

where $T = X + Y$ and $c_\alpha(T)$, $\gamma_\alpha(T)$ are determined by the conditional distribution of Y given T under the null hypothesis, namely, $(Y|T) \sim$ Hypergiometric$(T, m+n, n)$; i.e. the probability of drawing $Y = y$ white balls in T draws without replacement from an urn containing m black balls and n white balls.
B. The "Fisher exact test" is the conservative version of the above test which takes $\gamma(T) = 1$ and chooses $c_\alpha(T)$ as small as possible to still have $E\{\phi(X,Y)|T\} \leq \alpha$. This turns out to be quite conservative. A less conservative approach is to use the finite-sampling normal theory approximation to obtain an approximate critical point. Explain how to use the W-W-N-H CLT to obtain this approximate critical point.