Statistics 583, Problem Set 5 Solutions

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1. A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$	.54	.08	.12	.26
$p_1(x)$	.22	.16	.36	.26

(a) Find a most powerful test of size $\alpha = .15$ for testing p_0 versus p_1 and determine its power.
(b) Find a test ϕ which minimizes the Bayes risk with $0 - 1$ loss and prior distribution $(\lambda, 1-\lambda) = (2/5, 3/5)$; i.e. a test ϕ which minimizes $\lambda a + (1-\lambda)b$ where $a = E_0\phi$ and $b = E_1(1-\phi)$.

Solution:

x	1	2	3	4	Total
$p_0(x)$	.54	.08	.12	.26	1
$p_1(x)$	.22	.16	.36	.26	1
$(p1/p_0)(x)$	11/27	2	3	1	
$p_1 - (2/3)p_0(x)$	$-.14$	.32/3	.28	.26/3	
$\sqrt{p_0 p_1}$	0.345	0.113	0.208	0.26	.926
$\lvert p_0 - p_1 \rvert$	0.32	0.08	0.24	0.0	.64

(a) From the augmented table we see that $p_1(x)/p_0(x) = 11/27, 2, 3, 1$, according as $x = 1, 2, 3, 4$, so a MP test of size $\alpha = .2$ is given by

$$\phi(x) = \begin{cases} 1, & \text{if } x = 3, \\ (3/8), & \text{if } x = 2, \\ 0, & \text{if } x = 1 \text{ or } 4. \end{cases}$$

Then

$$E_0\phi(X) = P_0(X=3) + (3/8)P_0(X=2) = .12 + (3/8)(.08) = .12 + .03 = .15$$

while

$$\text{Power} = E_1\phi(X) = P_1(X=3) + (3/8)P_1(X=2) = .36 + (3/8)(.16) = .36 + 3(.02) = .42.$$

(b) To find a test ϕ which minimizes $\lambda a + (1-\lambda)b = \mathcal{R}(\lambda, \phi)$, we compute

$$\begin{aligned}\mathcal{R}(\lambda, \phi) &= \lambda E_0 \phi(X) + (1-\lambda) E_1 (1 - \phi(X)) \\ &= 1 - \lambda - \int \phi(x) \{(1-\lambda) p_1(x) - \lambda p_0(x)\} \, d\mu(x).\end{aligned}$$

From this last expression we see that the Bayes risk is minimized by choosing ϕ to be 1 when $(1-\lambda)p_1 - \lambda p_0 > 0$ and 0 otherwise. That is,

$$\phi(x) = \begin{cases} 1, & \text{if } p_1(x) > \lambda p_0(x)/(1-\lambda), \\ 0, & \text{if } p_1(x) < \lambda p_0(x)/(1-\lambda). \end{cases}$$

From the last row of the augmented table above we see that for $(\lambda, 1-\lambda) = (2/5, 3/5)$ the resulting $\phi(x) = 1_{\{2,3,4\}}(x)$. Then $a = E_0\phi(X) = .46$ and $b = E_1(1-\phi(X)) = .22$. Hence we have $(\lambda a + (1-\lambda)b)_{min} = (.92 + .66)/5 = 1.58/5 < .32$. (Note that for the NP rule we have $\lambda a_{NP} + (1-\lambda)b_{NP} = (2/5)(.15) + (3/5)(.58) = (.30 + 1.74)/5 = 2.04/5 > .40$ and $(1/2)\lambda a_{NP} + (1/2)b_{NP} = (1/2)(.15) + (1/2)(.58) = .73/2$.

2. Continuation of problem 1. (a) For P_0 and P_1 as given in problem 1, compute $d_{TV}(P_0, P_1)$, $H(P_0, P_1)$, and the affinity $\rho(P_0, P_1) = \int \sqrt{p_0 p_1} d\mu$. (b) For the product laws P_{0n} and P_{1n} (corresponding to observation of $X_1, \ldots, X_n$ i.i.d. P_0 or P_1 respectively) compute $\rho(P_{0n}, P_{1n})$ and $H(P_{0n}, P_{1n})$ for $n = 20,\ 50,\ 100$. What does this imply about the test, ϕ_n say, based on $X_1, \ldots, X_n$ which minimizes the sum of risks?

   Solution: (a) From the augmented table we see that

$$V(P_0, P_1) \equiv (1/2) \sum_{x=1}^{4} |p_0(x) - p_1(x)| = .64/2 = .32.$$

We also see that $\rho(P_0, P_1) = \sum_{x=1}^4 \sqrt{p_0(x) p_1(x)} = .92567$, and hence $H^2(P_0, P_1) = .0743\ldots$, while $H(P_0, P_1) = 0.272\ldots$.
(b) Now

$$\rho(P_{0n}, P_{1n}) = \rho(P_0^n, P_1^n) = \rho(P_0, P_1)^n$$

so that

$$H(P_{0,n}, P_{1,n}) = \sqrt{1 - \rho(P_{0,n}, P_{1,n})} = \sqrt{1 - \rho(P_0, P_1)^n}.$$

In particular for $n \in \{20, 50, 100\}$ we find the following table for values of $\rho(P_{0n}, P_{1n})$ and $H(P_{0,n}, P_{1,n})$.

n	20	50	100
$\rho(P_{0,n}, P_{1n})$	0.213305	0.0210137	0.000441577
$H(P_{0,n}, P_{1,n})$	0.886958	0.989437	0.999779