

Statistics 583, Problem Set 6

Wellner; 5/27/2015

Reading: Wasserman, Chapter 6, pages 125-144; Chapter 5, pages 61-95.

Van der Vaart, Chapter 24, pages 341-348.

Due: Wednesday, June 3, 2011

Reminder: Final exam: Monday, June 8, 8:30-10:30 AM

- Verify the variance part of Theorem 6.9, Wasserman, page 129.
 - Verify the proof of (6.12) - (6.14) in Theorem 6.11, Wasserman, page 129.
A question about the proof: Is the $o(1/n)$ term on page 142, line -8, correct? Does this have any effect on (6.12)?
 - Can you find the limiting distribution of $n^{1/3}(\hat{f}_n(x) - f(x))$ under hypotheses similar to those used in (a corrected version of) Theorem 6.11?
- Consider the kernel density estimator defined in (6.26), Wasserman, page 132. Show that if the density f and the kernel k satisfy the hypotheses of Wasserman's theorem 6.28, page 133, and $h = h_n$ satisfies the hypotheses of Theorem 6.27, then for fixed $x \in \mathbb{R}$,

$$\sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) \rightarrow_d N\left(0, f(x) \int k^2(x)dx\right).$$

- Under what restriction on h_n does it follow (from (a) together with further analysis of the bias) that

$$\sqrt{nh_n}(\hat{f}_n(x) - f(x)) \rightarrow_d N\left(0, f(x) \int k^2(x)dx\right)?$$

- If $h_n = cn^{-1/5}$ and the hypotheses of (a) hold, find the limiting distribution of $\sqrt{nh_n}(\hat{f}_n(x) - f(x))$.
- Under the same assumptions as in (c), find the limiting distribution of $\sqrt{nh_n}(\sqrt{\hat{f}_n(x)} - \sqrt{f(x)})$.
- Suppose that $x, y \in \mathbb{R}$ with $x < y$. Find the joint limiting distribution of $(\sqrt{nh_n}(\hat{f}_n(x) - f(x)), \sqrt{nh_n}(\hat{f}_n(y) - f(y)))$ under the assumptions in (b) and (c).

- Wasserman, problem 6.9.3, page 143.
 - Does (6.35) on Wasserman's page 136 hold? Give a formula that is more precise.

4. Suppose that X_1, \dots, X_n are i.i.d. $p \in \mathcal{P}$ where \mathcal{P} is some class of densities on some subset $\mathcal{X} \subset \mathbb{R}^d$. (Thus $p(x) \geq 0$ for all $x \in \mathcal{X}$ and $\int_{\mathcal{X}} p(x) dx = 1$.) Let \mathcal{M} denote the set of all functions p on \mathcal{X} with $p(x) \geq 0$ (without imposing $\int_{\mathcal{X}} p(x) dx = 1$). Show that maximizing

$$\begin{aligned} \tilde{l}_n(p) &\equiv \mathbb{P}_n(\log f(\cdot)) - \left(\int_{\mathcal{X}} p(y) dy - 1 \right) \\ &= n^{-1} \sum_{i=1}^n \log p(X_i) - \left(\int_{\mathcal{X}} p(y) dy - 1 \right) \end{aligned}$$

over \mathcal{M} yields the maximum of

$$l_n(p) \equiv \mathbb{P}_n(\log f(\cdot))$$

over \mathcal{P} . More exactly, show that for arbitrary $p \in \mathcal{M}$ we have $\tilde{l}_n(p) \leq l_n(p/c)$ where $c = \int_{\mathcal{X}} p(x) dx$ with equality if and only if $c = 1$. [This is the rationale for the first display in Wasserman's section 6.4, page 137.]

5. **Bonus optional problem:** Show that $\int k^2(z) dz$ is minimized over kernels k with $\int z^2 k(z) dz = 1 = \int k(z) dz$ by the Epanechnikov kernel

$$k(z) = \frac{3}{4\sqrt{5}} (1 - z^2/5) 1_{\{|z| \leq \sqrt{5}\}}.$$

Hint: see Silverman (1986), pages 41-42.]

6. **Bonus optional problem.**

- (a) Wasserman, problem 6.9.8, page 143. Is the second equality in Wasserman's (6.45) on page 138 correct?
 (b) Verify the formula (6.16) of Wasserman, page 129.