Statistics 583, Problem Set 3

Wellner; 4/22/2015

Reading: Chapter 7, sections 7.1- 7.4;
Wasserman, Chapters 1-2, pages 1-24.
Van der Vaart, Chapter 20, pages 291-303.

Due: Wednesday, April 29, 2015
Reminder: Midterm Exam, Wednesday, 6 May.
Reminder: Second make-up lecture, Wednesday, 29 April, 9:30 AM, GUG 204.

1. (See also van der Vaart (1998), page 303, problem 3.) For distribution functions F on R^+ and $t_0 > 0$, consider the functional $T(F) = \Lambda(t_0) \equiv \int_0^{t_0} \frac{1}{1-F_-} dF$, the *cumulative hazard function* corresponding to F at t_0.
   (a) Find the influence function of $T(F)$.
   (b) What does this mean about asymptotic normality of the natural estimator $T(\mathbb{F}_n)$ of $T(F)$?
   (c) Can you prove asymptotic normality of $T(\mathbb{F}_n)$ directly?

2. Let F be a distribution function on $\mathbb{R}^2$ with finite second moments, and let $\rho(F)$ be the correlation coefficient
$$\rho(F) = \frac{Cov_F(X,Y)}{\sqrt{Var_F(X) Var_F(Y)}} .$$
   Assume that $|\rho(F)| < 1$.
   (a) Give an example of a sequence of bivariate distributions $\{F_n\}$ satisfying $F_n \rightarrow_d F$, but $\rho(F_n) \rightarrow 1 \neq \rho(F)$.
   (b) Find a collection $\mathcal{F}$ of distribution functions on $\mathbb{R}^2$ so that ρ is weakly continuous on $\mathcal{F}$.

3. Exercise 3.8.1, Wasserman, page 39. [Hint: the formula given by Wasserman, page 29, is not correct.] Under what additional hypotheses can we establish $\sqrt{n}(T(\mathbb{F}_n) - T(F)) \rightarrow N(0, E_F \psi_F^2(X))$? (Here my ψ_F equals Wasserman's L_F.)

4. (a) Exercise 2.7.9, Wasserman, page 24.
   (b) What additional hypotheses are needed to show that $\sqrt{n}(T(\mathbb{F}_n) - T(F))$ is asymptotically normal for this particular functional $T(F)$?
   Reminder: This exercise gives the same result as we derived last Fall in Stat 581.

5. **Optional bonus problem 1:** Consider the collection $\mathcal{F}_0$ of distribution functions F on R^+ with $0 < E_F X < \infty$ and $E_F X^2 < \infty$. Let $T(F) \equiv \sigma(F)/\mu(F)$ for $F \in \mathcal{F}_0$ where $\sigma^2(F) = Var_F(X)$ and $\mu(F) = E_F(X)$. This is the *coefficient of variation of* F. Find the influence function of $T(F)$.

6. **Optional bonus problem 2:** Suppose that $\mathcal{F}_+$ is the class of distribution functions F on $\mathbb{R}^+$ with mean $\mu_F = E_F X < \infty$, and consider the functional $T(F)$ defined for a fixed $x_0 \in R^+$ by

$$T(F) \equiv e_F(x_0) \equiv E_F(X - x_0 | X > x_0) = \frac{\int_{x_0}^{\infty}(1 - F(t))dt}{1 - F(x_0)} .$$

This functional is the *mean residual life* functional.
(a) For what collection of df's F_0 is T weakly continuous at F_0? For what collection of df's F_0 is T continuous at F_0 with respect to the Kolmogorov metric?
(b) Find the influence function of $T(F)$.
(c) Can you prove asymptotic normality of $T(\mathbb{F}_n)$ directly?