Statistics 583, Problem Set 2 Solutions

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1. (Problem 10, page 249, Ferguson, MS) Let $\Theta = \{(\Delta, \pi_1, \ldots, \pi_n) : \ \Delta \geq 0, \ \pi = (\pi_1, \ldots, \pi_n)$ is a permutation of $\{1, \ldots, n\}\}$, and let the distribution of $X_1, \ldots, X_n$ given $\theta = (\Delta, \pi_1, \ldots, \pi_n)$ be as independent random variables with gamma distributions, $X_i \sim \text{Gamma}(\alpha, \beta^{-1}\exp(-\Delta b_{\pi_i}))$ where $\alpha > 0$, $\beta > 0$, and $b_1, \ldots, b_n$ are known real numbers with $\sum_1^n b_i > 0$. Consider testing the hypothesis $H : \Delta = 0$ versus the alternative $K : \Delta > 0$. (This is a Gamma-regression model with covariates or predictors b_i in which the relationship between the responses X_i and the covariates b_i have become scrambled or mixed up: we unfortunately don't know the right pairing of X_i and b_i, but we do know that some permutation of the b_i's is correct. Note that problem 11 in Ferguson, MS, gives a more realistic version of the problem in which β is also unknown. This is a version of a "broken sample" or "record linkage" problem; see e.g. Bai and Hsing, PTRF, 2005.)
(a) Show that this problem is invariant under the group of permutations of $(X_1, \ldots, X_n)$, and that the distribution of the maximal invariant $(Y_1, \ldots, Y_n) \equiv (X_{(1)}, \ldots, X_{(n)})$(the order statistics) has density

$$f_{\underline{Y}}(\underline{y}|\Delta) = \frac{(\prod_1^n y_i)^{\alpha-1}\exp(-\alpha\Delta\sum_1^n b_i)}{\Gamma(\alpha)^n\beta^{n\alpha}} \sum_{\pi\in\Pi} \exp\left\{-\frac{1}{\beta}\sum_{i=1}^n y_i \exp(-\Delta b_{\pi_i})\right\}$$

for $y_1 < \cdots < y_n$ and zero elsewhere where $\sum_{\pi\in\Pi}$ denotes the sum over all permutations π of $\{1, \ldots, n\}$.
(b) Show that the locally best invariant test of H versus K (i.e. the test which maximizes the slope of the power function at the null hypothesis) is to reject H when $\sum_{i=1}^n X_i$ is too large.

Solution: Let G be the permutation group

$$G = \{g : \ g(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)}), \ \pi \in \Pi\}.$$

Then, if $\underline{X} \sim P_\theta$, for $g = g_{\pi'} \in G$, $g(\underline{X}) \sim P_{\overline{g}(\theta)}$ with $\overline{g}(\theta) = (\Delta, \pi \circ \pi') = (\Delta, (\pi_{\pi'(1)}, \ldots, \pi_{\pi'(n)}))$. Thus the hypotheses are invariant under G. The order statistics are a $G-$MI. But of course the X_i's are *not* identically distributed. The joint density of the X_i's is given by

$$\begin{aligned} f(\underline{x}; \Delta, \pi) &= \prod_{i=1}^n \frac{x_i^{\alpha-1}e^{-\alpha\Delta b_{\pi(i)}}}{\Gamma(\alpha)\beta^\alpha}\exp(-\beta^{-1}e^{-\Delta b_{\pi(i)}}x_i) \\ &= \frac{\exp(-\Delta\sum_{j=1}^n b_j)(\prod_{i=1}^n x_i)^{\alpha-1}}{\Gamma(\alpha)^n\beta^{n\alpha}} \exp\left(-\beta^{-1}\sum_{i=1}^n x_i e^{-\Delta b_{\pi(i)}}\right). \end{aligned}$$

If $X_{(\cdot)} = (X_{(1)}, \ldots, X_{(n)})$ denotes the order statistics, then by problem 3(c) of problem set #9, Statistics 582,

$$
\begin{aligned}
& f_{X_{(\cdot)}}(\underline{x}_{(\cdot)}; \Delta, \pi) \\
&= \sum_{\pi' \in \Pi} f(\pi' \underline{x}; \Delta, \pi) \\
&= \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \exp(-\alpha\Delta \sum b_j) \sum_{\pi' \in \Pi} \prod_{i=1}^n x_{\pi'(i)}^{\alpha-1} \exp\left\{ -\frac{1}{\beta} \sum_{i=1}^n x_{(\pi'(i))} \exp(-\Delta b_{\pi_i}) \right\} \\
&= \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \exp(-\alpha\Delta \sum b_j) \prod_{i=1}^n x_{(i)}^{\alpha-1} \sum_{\pi^* \in \Pi} \exp\left\{ -\frac{1}{\beta} \sum_{i=1}^n x_{(i)} \exp(-\Delta b_{\pi_i^*}) \right\}
\end{aligned}
$$

with $\pi^* \equiv (\pi')^{-1} \circ \pi$. Note that this distribution depends only on the $\overline{G}-$MI Δ (and not on π).

(b) Now

$$
\begin{aligned}
l(\Delta | \underline{X}_{(\cdot)}) &\equiv \log f_{X_{(\cdot)}}(\underline{X}(\cdot); \Delta) \\
&= \log \left\{ \sum_{\pi^* \in \Pi} \exp\left\{ -\frac{1}{\beta} \sum_{i=1}^n X_{(i)} \exp(-\Delta b_{\pi^*(i)}) \right\} \right\} \\
&\quad - \alpha\Delta \sum_{i=1}^n b_i + \text{constant in} \quad \Delta
\end{aligned}
$$

so that

$$
\begin{aligned}
\dot{\mathbf{l}}_{\Delta}(\underline{X}_{(\cdot)}; \Delta = 0) &= \frac{1}{\sum_{\pi^*} \exp(\cdots)} \Big|_{\Delta=0} \\
&\quad \cdot \sum_{\pi^* \in \Pi} \exp(\cdots) \Big|_{\Delta=0} \left\{ \frac{1}{\beta} \sum_{i=1}^n X_{(i)} \exp(-\Delta b_{\pi^*(i)})(-b_{\pi^*(i)}) \Big|_{\Delta=0} \right\} \\
&\quad -\alpha \sum_{j=1}^n b_j \\
&= \frac{1}{\beta n!} \sum_{\pi^*} \left\{ \sum_{i=1}^n X_{(i)} b_{\pi^*(i)} \right\} - \alpha \sum_{i=1}^n b_i \\
&= \frac{1}{\beta n!} \sum_{i=1}^n X_{(i)} \sum_{\pi^*} b_{\pi^*(i)} - \alpha \sum_{i=1}^n b_i \\
&= \frac{1}{\beta} \bar{b} \sum_{i=1}^n X_{(i)} - \alpha n \bar{b} = \frac{n\bar{b}}{\beta} (\overline{X} - \alpha\beta),
\end{aligned}
$$

and hence the locally MP invariant test rejects for large values of $\sum_{i=1}^n X_i$.

For an extension to unknown β and to testing $H_0 : \Delta = 0$ versus $K : \Delta \neq 0$, see Ferguson, problem 11, page 249. In this version of the problem $\sum_1^n b_i = 0$ *is* assumed.

2. Suppose that $X_{i,j} \sim N(\mu_i, \sigma^2)$ for $j = 1, \ldots, n_i$, $i = 1, \ldots, k$, are all independent. Consider testing $H : \mu_1 = \ldots = \mu_k$ versus $K : \mu_i \neq \mu_j$ for some $i \neq j$.
(a) Relate this classical one-way ANOVA problem to the canonical form of the linear model.
(b) Can you relate the group assumed for the canonical form of the model to the particular form of the linear model assumed here? Is there an induced group on the $X_{i,j}$'s?
(c) Find the UMP G-invariant test of H versus K and give the form of the power function of your test.
(d) For the case $k = 2$, there is a UMP-unbiased test of H versus K given by the (two-sided) two sample $t-$test. Consider the special case $k = 3$. Does a UMP-unbiased test exist?

Solution: (a) Here $\xi = A\mu$ where A is an $n \times k$ matrix consisting of a first column $\underline{a}_1$ with n_1 ones and $n - n_1$ zeros, a second column $\underline{a}_2$ with n_1 zeros followed by n_2 ones, followd by $n - n_1 - n_2$ zeros, and so forth, ending with a $k-$th column $\underline{a}_k$ with $n - n_k$ zeros followed by n_k ones. The restriction $\mu_1 = \mu_2 = \cdots = \mu_k$ can be represented by the $(k-1) \times k$ matrix given by

$$B = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdots & & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Now if T is a matrix with last $n - k = \sum_{i=1}^k (n_i - 1)$ rows orthogonal to span$[\underline{a}_1, \ldots, \underline{a}_k]$, and first $k-1$ rows orthogonal to the $n-$vector $\underline{1} = (1, \ldots, 1)^T$, and $k-$th row equal to $n^{-1/2}\underline{1}$, then $Z = TX$ and $\eta = T\xi$ yields the linear model in its canonical form $Z \sim N_n(\eta, \sigma^2 I)$ with $\eta_{k+1} = \cdots = \eta_n = 0$, and with $\eta_1 = \cdots \eta_{k-1} = 0$ under the null hypothesis H.
(b) There is an induced group G^* on the $X_{i,j}$'s defined by $gz = g(Tx) = T(g^* x)$, or $g \circ T = T \circ g*$, or $g^*(x) = T'(gz)$ where T' is the transpose of T This is computable explicitly once the orthogonal matrix T has been determined. Thus, for example,

$$g_1^*(x) = T'(g_1 z) = T'(z_1, \ldots, z_k, z_{r+1} + \Delta_{r+1}, \ldots, z_k + \Delta_k, z_{k+1}, \ldots, z_n).$$

(c) Now $\xi_{i,j} = EX_{i,j} = \mu_i$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$, so

$$\sum_{i=1}^{k} \sum_{j=}^{n_i} (X_{i,j} - \mu_i)^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{i,j} - X_{i,\cdot})^2 + \sum_{i=1}^{k} n_i (X_{i,\cdot} - \mu_i)^2$$

where $X_{i,\cdot} = n_i^{-1} \sum_{j=1}^{n_i} X_{i,j}$ and hence $\hat{\xi}_{i,j} = \hat{\mu}_i = X_{i,\cdot}$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$. Furthermore,

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{i,j} - \mu)^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{i,j} - X_{\cdot,\cdot})^2 + n(X_{\cdot,\cdot} - \mu)^2$$

where $X_{\cdot,\cdot} = \sum \sum X_{i,j}/n$, $n = n_1 + \cdots + n_k$, and hence $\hat{\hat{\xi}}_{i,j} = X_{\cdot,\cdot}$ for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$. Thus the test becomes: reject $H: \ \mu_1 = \cdots = \mu_k$ (versus $K : \mu_i \neq \mu_j$ for some $i \neq j$) if

$$F \equiv \frac{\sum_{i=1}^{k} n_i (X_{i,\cdot} - X_{\cdot,\cdot})^2/(k-1)}{\sum \sum (X_{i,j} - X_{i,\cdot})^2/(n-k)} > F_{k-1,n-k,\alpha}.$$

The non-centrality parameter is

$$\delta = \frac{1}{\sigma^2} \sum_{i=1}^{k} n_i (\mu_i - \mu.)^2$$

where $\mu. = n^{-1} \sum_{i=1}^{k} n_i \mu_i$. The power function of this test is given by

$$\beta_\phi(\delta) \quad = \quad P_\delta(F > F_{k-1,n-k,\alpha}) = P(F_{k-1,n-k}(\delta) > F_{k-1,n-k,\alpha})$$

where $F_{k-1,n-k}(\delta)$ denotes a non-central F random variables with degrees of freedom $k-1$ and $n-k$ respectively, and non-centrality parameter δ.

(d) Let $k = 3$ and consider testing $\mu_1 = \mu_2 = \mu_3$. Then the density of the data is given by

$$\begin{aligned} p(\underline{x}; \underline{\mu}, \sigma^2) &= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{j=1}^{n_1} (x_{1,j} - \mu_1)^2 + \sum_{j=1}^{n_2} (x_{2,j} - \mu_2)^2 + \sum_{j=1}^{n_3} (x_{3,j} - \mu_3)^2 \right) \right) \\ &= C(\underline{\mu}, \sigma^2) \exp\left(\sum_{i=1}^{3} \frac{\mu_i}{\sigma^2} \sum_{j=1}^{n_i} x_{i,j} - \frac{1}{2\sigma^2} \sum \sum x_{i,j}^2 \right) \qquad (1) \\ &= C(\underline{\mu}, \sigma^2) \exp\left(\sum_{i=1}^{3} \frac{n_i \mu_i}{\sigma^2} x_{i,\cdot} - \frac{1}{2\sigma^2} \sum \sum x_{i,j}^2 \right) \qquad (2) \end{aligned}$$

When $\mu_1 = \mu_2 = \mu_3 \equiv \mu$, this reduces to

$$C'(\mu, \sigma^2) \exp\left(\frac{\mu}{\sigma^2} \sum_{i=1}^{3} \sum_{j=1}^{n_i} x_{i,j} - \frac{1}{2\sigma^2} \sum \sum x_{i,j}^2 \right),$$

so it is natural to condition on $T(\underline{X}) = (\sum\sum X_{i,j}, \sum\sum X_{i,j}^2)$, and to rewrite the density in (2) as

$$C(\underline{\mu}, \sigma^2) \exp\left(\sum_{i=1}^{3} \frac{n_i(\mu_i - \mu.)}{\sigma^2} x_{i,\cdot} + \frac{\mu.}{\sigma^2} \sum_{i=1}^{3} n_i x_{i,\cdot} - \frac{1}{2\sigma^2} \sum \sum x_{i,j}^2 \right)$$

for any choice of a real number $\mu.$; e.g. $\mu. = \sum_{i=1}^{3} n_i \mu_i / n$ or $\mu. = \mu_1$. Conditioning on the sufficient statistics for the null hypothesis reduces this to an exponential family involving just the terms

$$\sum_{i=1}^{3} \frac{n_i(\mu_i - \mu.)}{\sigma^2} x_{i,\cdot},$$

and choosing $\mu. = \mu_1$ reduces this to an exponential family involving the terms

$$\sum_{i=2}^{3} \frac{n_i(\mu_i - \mu_1)}{\sigma^2} x_{i,\cdot},$$

but it is not possible to reduce the current hypothesis testing problem, even in its conditioned form, to a one-dimensional problem. Hence the theory of UMP unbiased tests which we developed in section 6.2 does not apply.

3. Suppose that X_{ijk}, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ satisfy the general linear model with $\xi_{ijk} = \xi + \mu_i + \eta_j + \delta_{ij}$ where $\sum_i \mu_i = 0$, $\sum_j \eta_j = 0$, $\sum_j \delta_{ij} = 0$ for all i, and $\sum_i \delta_{ij} = 0$ for all j. (δ_{ij} is called the interaction effect of the ith row and the jth column.)
(a) Show that

$$\begin{aligned} S^2 &= \sum\sum\sum (X_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \\ &= \sum\sum\sum (X_{ijk} - \overline{X}_{ij.})^2 \\ &\quad + \sum\sum\sum (\overline{X}_{ij.} - \overline{X}_{i..} - \overline{X}_{.j.} + \overline{X}_{...} - \delta_{ij})^2 \\ &\quad + \sum\sum\sum (\overline{X}_{i..} - \overline{X}_{...} - \mu_i)^2 + \sum\sum\sum (\overline{X}_{.j.} - \overline{X}_{...} - \eta_j)^2 \\ &\quad + \sum\sum\sum (\overline{X}_{...} - \xi)^2 \end{aligned}$$

where $\overline{X}_{ij\cdot} = \sum_k X_{ijk}/K$, and so on.
(b) Find the UMP invariant test of the hypothesis of no row effect $H_0 : \mu_1 = \ldots = \mu_I = 0$. What is the distribution of the test statistic under the general linear hypothesis – including the noncentrality parameter?
(c) Find the UMP invariant test of the hypothesis of no interaction effect $H_0 : \delta_{ij} = 0$ for all i, j. What is the distribution of the test statistic under the general linear hypothesis?

Solution: (a) We write $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \equiv \sum$, and note that

$$\begin{aligned}
S^(\xi) &= \sum (X_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \\
&= \sum (X_{ijk} - \overline{X}_{ij\cdot} + \overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdot\cdot\cdot} \\
&\qquad + \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot\cdot\cdot} + \overline{X}_{\cdot j\cdot} - \overline{X}_{\cdot\cdot\cdot} \\
&\qquad + \overline{X}_{\cdot\cdot\cdot} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \\
&= \sum (X_{ijk} - \overline{X}_{ij\cdot})^2 \\
&\quad + \sum (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdot\cdot\cdot} - \delta_{ij})^2 \\
&\quad + \sum (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot\cdot\cdot} - \mu_i)^2 + \sum (\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdot\cdot\cdot} - \eta_j)^2 \\
&\quad + \sum (\overline{X}_{\cdot\cdot\cdot} - \xi)^2
\end{aligned}$$

where the ten cross product terms all vanish because

$$\begin{aligned}
&\sum_i (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdot\cdot\cdot}) = 0, \quad \text{for all } j, \\
&\sum_j (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdot\cdot\cdot}) = 0, \quad \text{for all } i, \\
&\sum_i (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot\cdot\cdot}) = 0, \quad \text{for all } j, \\
&\sum_j (\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdot\cdot\cdot}) = 0, \quad \text{for all } i, \\
&\sum_k (\overline{X}_{ijk} - \overline{X}_{ij\cdot}) = 0, \quad \text{for all } i, j.
\end{aligned}$$

(b) It follows from (a) that the least squares estimators of ξ_{ijk} are given by

$\widehat{\xi}_{i,j,k} = \widehat{\xi} + \widehat{\mu}_i + \widehat{\eta}_j + \widehat{\delta}_{ij}$ where

$$\begin{aligned}
\widehat{\xi} &= \overline{X}_{\cdots}, \\
\widehat{\mu}_i &= \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots}, \\
\widehat{\eta}_j &= \overline{X}_{\cdot j\cdot} - \overline{X}_{\cdots}, \\
\widehat{\delta}_{ij} &= \overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots}.
\end{aligned}$$

Thus the residual sum of squares under the big model is $\sum(X_{ijk} - \overline{X}_{ij\cdot})^2$. On the other hand, under the hypothesis of no row effect, the least squares estimators $\hat{\hat{\xi}}_{ijk} = \hat{\hat{\xi}} + \hat{\mu}_i + \hat{\hat{\eta}}_j + \hat{\hat{\delta}}_{ij}$ of ξ_{ijk} are now given by $\hat{\hat{\xi}} = \widehat{\xi}$, $\hat{\hat{\eta}}_j = \widehat{\eta}_j$, and $\hat{\hat{\delta}}_{ij} = \widehat{\delta}_{ij}$, but $\hat{\mu}_i = 0$, and then the residual sum of squares is

$$S^2(\hat{\hat{\xi}}) = \sum(X_{ijk} - \overline{X}_{ij\cdot})^2 + \sum(\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots})^2.$$

Here $n = IJK$, $k = IJ$, and $r = I - 1$. Thus the F statistic for testing $H: \ \mu_1 = \cdots = \mu_i = 0$ is given by

$$F = \frac{\sum(\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots})^2/(I-1)}{\sum(X_{ijk} - \overline{X}_{ij\cdot})^2/(IJK - IJ)}.$$

which has an $F_{I-1,IJ(K-1)}(\delta^2)$ distribution where $\delta^2 = JK\sum_{i=1}^{I}\mu_i^2/\sigma^2$.

(c) The least squares estimators of ξ_{ijk} under the big model are exactly the same as in (b): $\widehat{\xi}_{i,j,k} = \widehat{\xi} + \widehat{\mu}_i + \widehat{\eta}_j + \widehat{\delta}_{ij}$, and the residual sum of squares under the big model is also exactly the same. Under the hypothesis H that all $\delta_{ij} = 0$, the least squares estimators $\hat{\hat{\xi}}_{ijk} = \hat{\hat{\xi}} + \hat{\mu}_i + \hat{\hat{\eta}}_j + \hat{\hat{\delta}}_{ij}$ of ξ_{ijk} are now given by $\hat{\hat{\xi}} = \widehat{\xi}$, $\hat{\hat{\eta}}_j = \widehat{\eta}_j$, and $\hat{\mu}_i = \widehat{\mu}_i$, but now $\hat{\hat{\delta}}_{ij} = 0$, and the residual sum of squares is

$$S^2(\hat{\hat{\xi}}) = \sum(X_{ijk} - \overline{X}_{ij\cdot})^2 + \sum(\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots})^2.$$

4. Consider a two-way classification X_{ij}, $i = 1, \ldots, I$, $j = 1, \ldots, J$ with the assumptions of the general linear hypothesis for which $EX_{ij} = \alpha + \beta z_i + \eta_j$, where α, β, and η_j are unknown parameters subject to the restriction $\sum \eta_j = 0$, and where z_i are known numbers for which $\sum_1^I z_i = 0$ and $\sum_1^I z_i^2 = 1$.
(a) Find the UMP invariant test of the hypothesis

$$H_0: \ \eta_1 = \eta_2 = \cdots = \eta_J = 0.$$

(b) What is the distribution of the test statistic under the general linear hypothesis?

Solution: Here $\xi = (\xi_{ij},\ i = 1, \ldots, I,\ j = 1, \ldots, J)$ with $\xi_{ij} = \alpha + \beta z_i + \eta_j$ where $\sum_i z_i = 0$, $\sum_1^I z_i^2 = 1$, and $\sum_j \eta_j = 0$. Thus the ξ_{ij}'s are in a $J+1$-dimensional subspace of R^n where $n = IJ$. The null hypothesis imposes $r = J-1$ restrictions, and under the null hypothesis the ξ_{ij}'s are in a $2-$dimensional subspace of R^n. It is straightforward to see that the unrestricted least squares estimators are given by

$$\widehat{\alpha} = \overline{X}_{\cdot\cdot} = \frac{1}{IJ}\sum_{ij} X_{ij},$$

$$\widehat{\beta} = \sum_{i=1}^{I} \overline{X}_{i\cdot} z_i, \qquad \text{where } \overline{X}_{i\cdot} = \frac{1}{J}\sum_j X_{ij},$$

$$\widehat{\eta}_j = \overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot}.$$

Under the null hypothesis that $\eta_1 = \eta_2 = \cdots = \eta_J = 0$,

$$\hat{\hat{\alpha}} = \widehat{\alpha} = \overline{X}_{\cdot\cdot},$$

$$\hat{\hat{\beta}} = \widehat{\beta} = \sum_{i=1}^{I} \overline{X}_{i\cdot} z_i,$$

as before, but now $\hat{\hat{\eta}}_j = 0,\ j = 1, \ldots, J$. Hence

$$\widehat{\xi}_{ij} - \hat{\hat{\xi}}_{ij} = \widehat{\eta}_j = \overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot}$$

and

$$\sum_{i,j} (\widehat{\xi}_{ij} - \hat{\hat{\xi}}_{ij})^2 = \sum (\overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot})^2 = I \sum_{j=1}^{J} (\overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot})^2$$

while

$$\sum_{i,j} (X_{ij} - \widehat{\xi}_{ij})^2 = \sum_{i,j} (X_{ij} - \overline{X}_{\cdot j} - \widehat{\beta} z_i)^2.$$

Thus the F statistic becomes

$$F = \frac{I \sum_{j=1}^{J} (\overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot})^2 / (J-1)}{\sum_{i,j} (X_{ij} - \overline{X}_{\cdot j} - \widehat{\beta} z_i)^2 / (IJ - J - 1)}.$$

Under the general linear hypothesis the noncentrality parameter is

$$\delta^2 = \frac{1}{\sigma^2} \sum_{i,j} \widehat{\xi}_{ij}(\underline{\xi}) - \hat{\hat{\xi}}(\underline{\xi}))^2 = \frac{I \sum_{j=1}^{J} \eta_j^2}{\sigma^2}.$$

Hence under the general linear hypothesis

$$F \sim F_{J-1, IJ-J-1}(\delta^2).$$