**Statistics 583, Problem Set 1 Solutions**

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1. Suppose that $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$ are independent. In Statistics 582 last quarter we derived the UMP unbiased (conditional) test of level $\alpha$ for testing $H : p_2 \le p_1$ versus $K : p_2 > p_1$. It involves rejecting $H$ if $Y > c(t)$ relative to the conditional (hypergeometric) distribution of $Y$ conditional on $T = X + Y = t$, or equivalently if

$$\frac{Y/t - (n/N)}{\sigma_N} > c'(t)$$

where $c'(T) \to_p z_\alpha \equiv \Phi^{-1}(1-\alpha)$ since

$$\frac{Y/t - (n/N)}{\sigma_N} \to_d Z \sim N(0,1)$$

conditionally on $T$ if $0 < \liminf(n/N) \le \limsup(n/N) < 1$. Here $\sigma_N^2 = (1 - (t-1)/(N-1))(n/N)(1-n/N)/t$.

(a) Show that

$$\frac{Y/t - (n/N)}{\sigma_N} = \frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n} - \frac{X}{m}\right)}{\sqrt{\frac{t}{N}\left(1 - \frac{t-1}{N-1}\right)}}.$$

(b) Use the result of (a) to show that

$$\frac{Y/T - (n/N)}{\sigma_N} \to_d Z \sim N(0,1)$$

unconditionally under $p_1 = p_2 \equiv p$ (even if $a = \liminf(n/N) < \limsup(n/N) = b$ with possibly $a = 0$ or $b = 1$). [Hint: prove it first under the assumption that $n/N \to \lambda \in [0,1]$, and then show that the limit is the same even if $n/N$ does not converge by considering subsequences.

(c) What is the (unconditional) limiting behavior of the test statistic $(Y/t - (n/N))/\sigma_N$ under local alternatives of the form $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$ assuming that $n/N \to \lambda \in (0,1)$?

(d) What does the result of (c) imply about the limiting power of the test under these alternatives?

**Solution:** (a) Note that

$$\begin{aligned}
\frac{Y}{T} - \frac{n}{N} &= \frac{1}{T}\left(Y - \frac{n}{N}T\right) = \frac{1}{T}\left(Y - \frac{n}{N}(X+Y)\right) \\
&= \frac{1}{T}\left(\frac{m}{N}Y - \frac{n}{N}X\right) = \frac{mn}{NT}\left(\frac{Y}{n} - \frac{X}{m}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}\frac{Y/t-(n/N)}{\sigma_N} &= \frac{Y/t-n/N}{\sqrt{(1-\frac{n-1}{N-1})\frac{n}{N}\frac{m}{N}\frac{1}{t}}} = \frac{\sqrt{t}(Y/t-n/N)}{\sqrt{(1-\frac{n-1}{N-1})\frac{n}{N}\frac{m}{N}}} \\ &= \frac{\frac{mn}{N}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{T(1-\frac{T-1}{N-1})\frac{mn}{N^2}}} = \frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}}\end{aligned}$$

as claimed.

(b) Now $(\sqrt{m}(X/m-p), \sqrt{n}(Y/n-p)) \rightarrow_d (Z_1, Z_2)$ where $Z_j \sim N(0, p(1-p))$, $j = 1, 2$ are independent. Thus from (a), under the assumption that $n/N \rightarrow \lambda \in [0,1]$,

$$\begin{aligned}V_N \equiv \frac{Y/T-(n/N)}{\sigma_N} &= \frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}} \\ &= \frac{\sqrt{\frac{m}{N}}\sqrt{n}(Y/n-p)-\sqrt{\frac{n}{N}}\sqrt{m}(X/m-p)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}} \\ &\rightarrow_d \frac{\sqrt{1-\lambda}Z_2-\sqrt{\lambda}Z_1}{\sqrt{p(1-p)}} \sim N(0,1)\end{aligned}$$

where we used $T/N = (m/N)(X/m) + (n/N)(Y/n) \rightarrow_p (1-\lambda)p + \lambda p = p$.

If $\lambda_N \equiv n/N \not\rightarrow$, then since $\lambda_N \in [0,1]$, for any initial subsequence $\{\lambda_{N'}\}$, there exists a further convergent subsequence $\{\lambda_{N''}\}$; i.e. $\lambda_{N''} \rightarrow$ some $\lambda \in [0,1]$. By the same argument as above, for this subsequence $V_{N''} \rightarrow_d Z \sim N(0,1)$. Since the limiting distribution is the same for any such initial subsequence $\{V_{N'}\}$, we conclude that the full sequence $\{V_N\}$ satisfies $V_N \rightarrow_d Z \sim N(0,1)$ under $p_1 = p_2 = p$. (This argument is completely analogous to the following fact concerning real numbers: a sequence $\{x_n\}$ of real numbers satisfies $x_n \rightarrow x$ if and only if each subsequence $\{x_{n'}\}$ contains a further subsequence $\{x_{n''}\}$ such that $x_{n''} \rightarrow x$. See Billingsley (1968), *Convergence of Probability Measures*, theorem 2.3, page 16.)

(c) Under local alternatives $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$, we have

$$\begin{aligned}\sqrt{n}(Y/n-p_1) &= \sqrt{n}(Y/n-p_1-c/\sqrt{N}) + \sqrt{n}c/\sqrt{N} \\ &= \sqrt{n}(Y/n-p_{2,N}) + c\sqrt{\frac{n}{N}} \\ &\rightarrow_d Z_2 + c\sqrt{\lambda} \sim N(c\sqrt{\lambda}, p_1(1-p_1))\end{aligned}$$

under the assumption that $\lambda_N = n/N \to \lambda$. Then it follows that $(\sqrt{m}(X/m - p_1), (\sqrt{n}(Y/n - p_1)) \to_d (Z_1, Z_2 + c\sqrt{\lambda})$ and hence

$$V_N \to_d \frac{\sqrt{1-\lambda}(Z_2 + c\sqrt{\lambda}) - \sqrt{\lambda} Z_1}{\sqrt{p_1(1-p_1)}} \sim N\left(c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}}\ ,\ 1\right).$$

(d) The limiting distribution under local alternatives found in (c) implies that the power of the test based on $V_N$ statisfies

$$\begin{aligned}
\lim_{N\to\infty} \beta((p_1, p_{2,N})) &= \lim_{N\to\infty} P_{(p_1,p_{2,N})}(V_N > z_\alpha) \\
&= P\left(Z + c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}} > z_\alpha\right) \\
&= 1 - \Phi\left(z_\alpha - c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}}\right).
\end{aligned}$$

2. Read TPE (Lehmann and Casella) pages 160 - 162 concerning the notion of *equivariance* of an estimator $\delta = \delta(X)$ under a group of transformations $G$. Relate this to *invariance* of a (test) function $\phi$ under a group of transformations $G$. Illustrate equivariance with two examples.

   **Solution:** Let $X \sim P_\theta$ for $\theta \in \Theta$, and suppose that we want to estimate some function $h(\theta) \in \mathcal{H}$. For a group of transformations $G$ on the sample space $\mathcal{X}$, we typically induce a group $\overline{G}$ on the parameter space $\Theta$ via the correspondence $gX \sim P_{\overline{g}\theta}$. Suppose, moreover, that $\overline{G}$ induces a group $G^*$ on $\mathcal{H}$ via $h(\overline{g}\theta) = g^* h(\theta)$. If $\delta : \mathcal{X} \to \mathcal{H}$ yields an estimator $\delta(X)$ of $h(\theta)$, then we expect to use $g^* \circ \delta(X)$ or $\delta(gX)$ to estimate $h(\overline{g}\theta)$. Thus equivariance is just the requirement that $g^* \circ \delta(X) = \delta(gX)$.

   It is fairly straightforward to relate this to the testing situation, in which case $h(\theta) = 1_{\Theta_K}(\theta)$ and the induced group $G^*$ reduces to the trivial group $G = \{e\}$.

   Here are two examples of equivariance in estimation:

   **Example 1. Location** Suppose that $X = (X_1, \ldots, X_n)$ where the $X_i$'s are i.i.d. $N(\theta, \sigma^2)$ where $\sigma^2 > 0$ is known. We want to estimate $h(\theta) = \theta$. If $G = \{g_c : g_c(x) = x + c\underline{1},\ c \in R\}$, then the induced group on the parameter space is $\overline{G} = \{\overline{g}_c : g_c(\theta) = \theta + c,\ c \in R\}$, and this is also the group $G^*$ in the discussion above. Note that for the usual estimator $\delta(X) = \overline{X} = n^{-1}\sum_1^n X_i$ we have

$$\delta(g_c X) = \overline{X} + c = g_c^*(\overline{X}),$$

   i.e. $\delta = \overline{X}$ is (location) equivariant.

**Example 2. Scale** Suppose that $X = (X_1, \ldots, X_n)$ where the $X_i$'s are i.i.d. $N(0, \theta^2)$ where $\theta > 0$ is unknown. We want to estimate $h(\theta) = \theta^2$. If $G = \{g_c : \ g_c(x) = cx, \ x \in R^n, \ c > 0\}$, then the induced group on the parameter space $\Theta = \{\theta : \ \theta > 0\}$ is $\overline{G} = \{\overline{g}_c : \ \overline{g}_c(\theta) = c\theta\}$, and in this case the group $G^* = \{g_c^* : \ g_c^*(h) = c^2 h : \ c > 0\}$ since $h(\overline{g}\theta) = (c\theta)^2 = c^2 h(\theta)$. For the natural (consistent) estimator $\delta(X) = S_X^2 = n^{-1}\sum_1^n (X_i - \overline{X})^2$ of $h(\theta) = \theta^2$, we have

$$\delta(g_c(X)) = c^2 S_X^2 = g_c^*(\delta(X)),$$

i.e. $\delta = S_X^2$ is (scale)-equivariant.

3. Let $X$ and $Y$ be independent exponential random variables with parameters $\lambda$ and $\mu$ respectively: thus $P(X > x) = \exp(-\lambda x)$ and $P(Y > y) = \exp(-\mu y)$ for $x, y \geq 0$. Let $\theta \equiv \lambda/\mu$.
(a) Show that the problem of testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$ is invariant under the group $G$ of transformations $g_c(x, y) = (cx, cy)$, $c > 0$.
(b) Find a maximal invariant.
(c) Find a UMP invariant test of size $\alpha$.
(d) Show that the problem of testing $H_0' : \theta = 1$ versus $H_1' : \theta \neq 1$ is invariant *in addition* under the transformation $g(x, y) = (y, x)$.
(e) Find a UMP invariant test of size $\alpha$.
(f) Find UMP invariant tests of the hypotheses in (a) and (d) when $X_1, \ldots, X_m$ are i.i.d. Exponential($\lambda$) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential($\mu$).

**Solution:** (a) Now $X, Y$ have joint density

$$p_{\lambda,\mu}(x, y) = \lambda e^{-\lambda x} \mu e^{-\mu y} 1_{[0,\infty)}(x) 1_{[0,\infty)}(y)$$

so that if $c > 0$, $g_c(X, Y) = (cX, cY) \sim p_{\lambda/c, \mu/c}(x, y)$ and hence $\overline{g}(\lambda, \mu) = (\lambda/c, \mu/c)$. Note that $\delta(\lambda, \mu) = \lambda/\mu = (\lambda/c)/(\mu/c) = \delta(\overline{g}(\lambda, \mu))$, so that the hypotheses $H : \delta \leq 1$ and $K : \ \delta > 1$ are invariant. $T(X, Y) = X/Y$ and $\delta(\lambda, \mu) = \lambda/\mu$ are maximmal invariants on the sample space and parameter space respectively, and since $2\mu Y \sim \chi_2^2$ and $2\lambda X \sim \chi_2^2$ are independent

$$T(X, Y) = \frac{2\lambda X}{2\mu Y}\frac{\mu}{\lambda} \sim \delta^{-1} F_{2,2}.$$

Since this family of distributions has monotone (decreasing) likelihood ratio, it follows that the UMP $G-$invariant test of $H$ versus $K$ rejects $H$ when $T < F_{2,2,\alpha}$ (where $P(F_{2,2} \leq F_{2,2,\alpha}) = \alpha$. or when $T^{-1} = Y/X > F_{2,2,1-\alpha} = (1-\alpha)/\alpha$.
(b) If $g(x, y) = (y, x)$ is also considered, then the transformation induced on the maximal invariant of A is given by

$$g(T) = T(g(X, Y)) = T(Y, X) = \frac{Y}{X} = \frac{1}{T(X, Y)}.$$

**Claim:** $T \vee T^{-1}$ is the maximal invariant with respect to this new group.
Proof. $T \vee T^{-1}$ is invariant since $T^{-1} \vee T = T \vee T^{-1}$; and $T \vee T^{-1}$ is maximal since $T \vee T^{-1} = T^* \vee T^{*-1}$ implies that either $T = T^*$ or $T = T^{*-1}$.

A corresponding maximal invariant on the parameter space is $\delta \vee \delta^{-1} = \frac{\lambda}{\mu} \vee \frac{\mu}{\lambda} \equiv \nu$, and the hypotheses $H : \delta = 1$ and $K : \delta \neq 1$ are clearly invariant. When expressed in terms of $\delta$ the hypotheses become $H : \delta = 1$ versus $K : \delta > 1$. It remains to show that the maximal invariant has monotone likelihood ratio (MLR).

By direct calculation, for $t > 1$,

$$1 - F_\eta(t) \quad = \quad P_{\lambda,\mu}(T \vee T^{-1} > t) = \frac{2 + \eta t}{1 + \eta t + t^2} \tag{1}$$

with $\eta = \delta + 1/\delta$. Hence $T \vee T^{-1}$ has density

$$f_\eta(t) = \frac{\eta(t^2+1) + 4t}{(t^2 + \eta t + 1)^2} 1_{[1,\infty)}(t).$$

so that for $\eta_1 < \eta_2$

$$\frac{f_{\eta_2}}{f_{\eta_1}}(t) = \left( \frac{t^2 + \eta_1 t + 1}{t^2 + \eta_2 t + 1} \right)^2 \frac{\eta_2(t^2+1) + 4t}{\eta_1(t^2+1) + 4t} \equiv g(t)^2 h(t)$$

where

$$g'(t) = \frac{(\eta_2 - \eta_1)(t^2 - 1)}{(t^2 + \eta_2 t + 1)^2} \geq 0 \qquad \text{for} \quad t \geq 1$$

and

$$h'(t) = \frac{(\eta_2 - \eta_1)(4t^2 - 1)}{(t^2 + \eta_2 t + 1)^2} \geq 0 \qquad \text{for} \quad t \geq 1.$$

Hence the distribution of $T \vee T^{-1}$ has MLR and the UMP $G-$invariant test rejects $H$ when $M \equiv T \vee T^{-1} > \frac{2-\alpha}{\alpha}$. Then, when $\delta = 1$, $\eta = 2$, and

$$P_{\eta=2}(M > (2-\alpha)/\alpha)) = \alpha.$$

[Proof of (**??**): Since $T \sim \delta^{-1} F_{2,2}$ where $P(F_{2,2} \leq x) = x/(1+x)$,

$$\begin{aligned} P(T \vee T^{-1} > t) &= P(T > t) + P(T < t^{-1}) = P(F_{2,2} > \delta t) + P(F_{2,2} < \delta/t) \\ &= \frac{1}{1 + \delta t} + \frac{\delta/t}{1 + \delta/t} \\ &= \frac{1/\delta}{1/\delta + t} + \frac{\delta}{\delta + t} \\ &= \frac{2 + (\delta + 1/\delta)t}{1 + (\delta + 1/\delta)t + t^2}.] \end{aligned}$$

(c) When $X_1, \ldots, X_m$ are i.i.d. exponential($\lambda$) and $Y_1, \ldots, Y_n$ are i.i.d. exponential($\mu$) respectively, and $G = \{g : g(\underline{x}, \underline{y}) = (c\underline{x}, c\underline{y}), c > 0\}$, then we reduce by sufficiency first: $(\sum_1^m X_i, \sum_1^n Y_j)$ is sufficient for $(\lambda, \mu)$. Then $T = \sum_1^m X_i / \sum_1^n Y_j$ is a maximal invariant with respect to $G^* = \{g^* : g^*(x, y) = (cx, cy),\ c > 0\}$ acting on the space of the sufficient statistic. Moreover,

$$T = \frac{\mu}{\lambda}\frac{m}{n}\frac{2\lambda \sum_1^m X_i/(2m)}{2\mu \sum_1^n Y_j/(2n)} \sim \frac{1}{\delta n/m} F_{2m,2n}$$

where $\delta \equiv \delta(\lambda, \mu) = \lambda/\mu$ is a maximal invariant with respect to $\overline{G}$ on the parameter space, and the density of $F_{2m,2n}$ is given by

$$f_{F_{2m,2n}}(x) = c_{m,n} \frac{x^{m-1}}{(1 + (m/n)x)^{m+n}} 1_{(0,\infty)}(x).$$

Thus the density of $T$ is given by

$$f_T(x, \delta) = \delta \tilde{c}_{m,n} \frac{(\delta x)^{m-1}}{(1 + \delta x)^{m+n}} 1_{(0,\infty)}(x)$$

which has monotone (decreasing) likelihood ratio in $M(x) = x$. Thus the UMP-$G^*$ invariant test rejects $H_0$ when $((n/m)T)^{-1} = (\sum_1^n Y_j/2n)/(\sum_1^m X_i/2m) > F_{2n,2m,\alpha}$ where $P(F_{2n,2m} > F_{2n,2m,\alpha}) = \alpha$.

For testing $H_0' : \theta = 1$ versus $H_1' : \theta \neq 1$, we find, much as in (b), that $\tilde{T} \equiv T \vee T^{-1}$ is a maximal invariant with respect to the induced group $G^* = G_2^* \oplus G_1^*$ on the space of the sufficient statistic. By a calculation similar to that of part (b),

$$\begin{aligned}
1 - F(t) &= P(T \vee T^{-1} > t) = P(T > t) + P(T < 1/t) \\
&= P(\delta^{-1}(m/n)F_{2m,2n} > t) + P(\delta^{-1}(m/n)F_{2m,2n} < 1/t) \\
&= P((m/n)F_{2m,2n} > \delta t) + P((m/n)F_{2m,2n} < \delta/t)
\end{aligned}$$

and hence

$$\begin{aligned}
f_{T \vee T^{-1}}(t) &= \delta \tilde{c}_{m,n} \frac{(\delta t)^{m-1}}{(1 + \delta t)^{m+n}} + \frac{\delta}{t^2} \tilde{c}_{m,n} \frac{(\delta/t)^{m-1}}{(1 + \delta/t)^{m+n}} \\
&= \tilde{c}_{m,n} \frac{t^{m-1}(\delta + t)^m (1 + t/\delta)^n + t^{n-1}(1 + \delta t)^m (1/\delta + t)^n}{(t^2 + (\delta + 1/\delta)t + 1)^{m+n}}.
\end{aligned}$$

It is not yet clear to me that this density depends only on $\eta = \delta + 1/\delta$, so I may be making a mistake somewhere.