# Statistics 583, Problem Set 9, Corrected

Wellner; 5/25/2011

**Reading:** Wasserman, Chapters 4-9, pages 43-223
**Due:** Wednesday, June 1, 2011
**Reminder:** Final exam: Monday, June 6, 8:30-10:30 AM

1. (a) Verify the variance part of Theorem 6.9, Wasserman, page 129.
   (b) Verify the proof of (6.12) - (6.14) in Theorem 6.11, Wasserman, page 129. A question about the proof: Is the $o(1/n)$ term on page 142, line -8, correct? Does this have any effect on (6.12)?
   (c) Can you find the limiting distribution of $n^{1/3}(\widehat{f}_n(x) - f(x))$ under hypotheses similar to those used in (a corrected version of) Theorem 6.11?

2. (a) Consider the kernel density estimator defined in (6.26), Wasserman, page 132. Show that if the density $f$ and the kernel $k$ satisfy the hypotheses of Wasserman's theorem 6.28, page 133, and $h = h_n$ satisfies the hypotheses of Theorem 6.27, then for fixed $x \in \mathbb{R}$,

$$\sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) \to_d N\left(0, f(x)\int k^2(x)dx\right).$$

   (b) Under what restriction on $h_n$ does it follow (from (a) together with further analysis of the bias) that

$$\sqrt{nh_n}(\hat{f}_n(x) - f(x)) \to_d N\left(0, f(x)\int k^2(x)dx\right)?$$

   (c) If $h_n = cn^{-1/5}$ and the hypotheses of (a) hold, find the limiting distribution of $\sqrt{nh_n}(\hat{f}_n(x) - f(x))$.
   (d) Under the same assumptions as in (c), find the limiting distribution of $\sqrt{nh_n}(\sqrt{\hat{f}_n(x)} - \sqrt{f(x)})$.
   (e) Suppose that $x, y \in \mathbb{R}$ with $x < y$. Find the joint limiting distribution of $(\sqrt{nh_n}(\hat{f}_n(x) - f(x)), \sqrt{nh_n}(\hat{f}_n(y) - f(y))$ under the assumptions in (b) and (c).

3. (a) Wasserman, problem 6.9.3, page 143.
   (b) Does (6.35) on Wasserman's page 136 hold? Give a formula that is more precise.

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. $p \in \mathcal{P}$ where $\mathcal{P}$ is some class of densities on some subset $\mathcal{X} \subset \mathbb{R}^d$. (Thus $p(x) \geq 0$ for all $x \in \mathcal{X}$ and $\int_{\mathcal{X}} p(x)dx = 1$.) Let $\mathcal{M}$ denote the set of all functions $p$ on $\mathcal{X}$ with $p(x) \geq 0$ (without imposing $\int_{\mathcal{X}} p(x)dx = 1$). Show that maximizing

$$\begin{aligned} \widetilde{l}_n(p) &\equiv \mathbb{P}_n(\log f(\cdot)) - \left( \int_{\mathcal{X}} p(y)dy - 1 \right) \\ &= n^{-1} \sum_{i=1}^{n} \log p(X_i) - \left( \int_{\mathcal{X}} p(y)dy - 1 \right) \end{aligned}$$

over $\mathcal{M}$ yields the maximum of

$$l_n(p) \equiv \mathbb{P}_n(\log f(\cdot))$$

over $\mathcal{P}$. More exactly, show that $\widetilde{l}_n(p/c) \geq l_n(p)$ where $c = \int_{\mathcal{X}} p(x)dx$ with equality if and only if $c = 1$. [This is the rationale for the first display in Wasserman's section 6.4, page 137.]

5. **Bonus optional problem.** In the context of the histogram estimator in problem 1 above, is there a choice of the bin width $h = 1/m$ that leads to the estimator being asymptotically Poisson rather than asymptotically Gaussian?

6. **Bonus optional problem.** (a) Wasserman, problem 6.9.8, page 143. Is the second equality in Wasserman's (6.45) on page 138 correct?
(b) Verify the formula (6.16) of Wasserman, page 129.