

Statistics 583, Problem Set 5

Wellner; 4/27/2011

Reading: Chapter 7, sections 7.4-7.6; Chapter 8, sections 8.1- 8.2;

Wasserman, Chapters 2-3, pages 13-41.

Due: Wednesday, May 4, 2011

Reminder: Midterm exam, Friday, May 6.

1. Exercise 3.8.1, Wasserman, page 39. [Hint: the formula given by Wasserman, page 29, is not correct.] Under what additional hypotheses can we establish $\sqrt{n}(T(\mathbb{F}_n) - T(F)) \rightarrow N(0, E_F \psi_F^2(X))$? (Here my ψ_F equals Wasserman's L_F .)

2. (a) Exercise 2.7.9, Wasserman, page 24.

(b) What additional hypotheses are needed to show that $\sqrt{n}(T(\mathbb{F}_n) - T(F))$ is asymptotically normal for this particular functional $T(F)$?

Reminder: This exercise gives the same result as we derived last Fall in Stat 581.

3. Suppose that we observe X_1, \dots, X_n i.i.d. P on $\mathbb{R}^+ = [0, \infty)$ and assume that $P \in \mathcal{P}_0 \equiv \{P_\theta : (dP_\theta/d\lambda) = p_\theta, \theta \in \Theta\}$ where $\theta = (\alpha, \beta) \in (0, \infty)^2$ and $p_\theta = p_{\alpha, \beta}$ is the Weibull density given by $p_\theta(x) = (\beta/\alpha)(x/\alpha)^{\beta-1} \exp(-(x/\alpha)^\beta) 1_{(0, \infty)}(x)$. From Lehmann and Casella, TPE, Example 6.6.1 (page 468) and problems 6.6.1 - 6.6.3 (page 509), we know that the maximum likelihood estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ exists and is unique if $0 < X_{(1)} < X_{(n)}$.

(a) If, in fact, $P \notin \mathcal{P}_0$, to what function of P do you expect $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ converges (in probability)? [Hint: use the development in example 6.6.1 of Lehmann and Casella, TPE, page 468, to show that the solution of the population version of the score equations (with sampling from $P \notin \mathcal{P}$) leads to $\alpha(P) = \{E_P(X^\beta)\}^{1/\beta}$ where β is the solution of

$$\frac{E_P(X^\beta \log X)}{E_P X^\beta} - \frac{1}{\beta} = E_P(\log X),$$

assuming that $E_P(X^\beta |\log X|) < \infty$.]

(b) Show heuristically that $\theta(P) = (\alpha(P), \beta(P))$ minimizes $K(P, P_\theta)$ over Θ .

(c) In particular, if P has “half-normal” density $p(x) = 2\phi(x)1_{(0, \infty)}(x)$ where ϕ is the standard normal density, find $(\alpha, \beta) = (\alpha(P), \beta(P))$ corresponding to the “best-fitting” member of the Weibull family $P_{(\alpha(P), \beta(P))}$. Plot both p and $p_{(\alpha(P), \beta(P))}$ as functions of x .

4. Suppose that $\hat{\theta}_n$ is the MLE for the Weibull family as in problem 3 above, and that $P \notin \mathcal{P}_0$. Heuristically we expect that

$$\sqrt{n}(\hat{\theta}_n - \theta(P)) \rightarrow_d N_2(0, \Sigma(P)) \quad (1)$$

for some covariance matrix $\Sigma = \Sigma(P)$ as $n \rightarrow \infty$.

- (a) What is the form of Σ that you expect in (1)?
- (b) What methods could be used to make these heuristics precise?

5. **Optional bonus problem:** Consider the class of all two-dimensional balls \mathcal{C} in \mathbb{R}^2 : i.e.

$$\mathcal{C} = \{C : C = \{x \in \mathbb{R}^2 : \|x - x_0\| \leq r\}, x_0 \in \mathbb{R}^2, r > 0\}.$$

(This is exactly the class \mathcal{A} of two-dimensional spheres described in Wasserman's Exercise 2.7.14, page 25, with my x_0, r being Wasserman's $(a, b), c$.)

- (a) Find the VC dimension of \mathcal{A} .
- (b) Suppose that X_1, \dots, X_n are i.i.d. P on \mathbb{R}^2 , and let \mathbb{P}_n be the empirical measure of the X_i 's. Use the Vapnik-Chervonenkis inequality given in Wasserman's theorem 2.41 to describe a (conservative) $1 - \alpha = .95$ simultaneous confidence set for all the probabilities $\{P(C) : C \in \mathcal{C}\}$.

6. **Optional bonus problem:** (Asymptotic equivalence of Huber's estimator and trimmed means.) Suppose that X_1, \dots, X_n are i.i.d. F with density f which is symmetric about some point θ_0 . Let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ denote the order statistics. For an integer k between 1 and $n/2$, let $\alpha = k/n$. Then the α -trimmed mean is

$$T_\alpha(\mathbb{F}_n) = \frac{1}{1 - 2\alpha} \int_\alpha^{1-\alpha} \mathbb{F}_n^{-1}(t) dt = \frac{1}{n - 2k} \sum_{i=k+1}^{n-k} X_{(i)}.$$

Show that under modest regularity conditions $T_\alpha(F) = T_{Hub}(F)$ and that $\sqrt{n}(T_\alpha(\mathbb{F}_n) - T_\alpha(F))$ has the same limiting distribution as does $\sqrt{n}(T_{Hub}(\mathbb{F}_n) - T_{Hub}(F))$ for Huber's estimator $T_{Hub}(\mathbb{F}_n)$ defined as the solution of

$$\int \psi_0(x - T_{Hub}(\mathbb{F}_n)) d\mathbb{F}_n(x) = 0$$

where

$$\psi_0(x) = \begin{cases} -a, & x < -a \\ x, & -a \leq x \leq a \\ a, & x \geq a. \end{cases}$$

with $1 - \alpha = F(a + \theta_0)$.