

Statistics 583, Problem Set 1

Wellner; 3/30/2011

Reading: Chapter 6, sections 6.3 and 6.4;

Lehmann and Romano, TSH, Chapters 6 and 7.

See also Ferguson, MS, Chapter 5, sections 5.6 and 5.7;

Due: Wednesday, April 6, 2011

1. Suppose that $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$ are independent. In Statistics 582 last quarter we derived the UMP unbiased (conditional) test of level α for testing $H : p_2 \leq p_1$ versus $K : p_2 > p_1$. It involves rejecting H if $Y > c(t)$ relative to the conditional (hypergeometric) distribution of Y conditional on $T = X + Y = t$, or equivalently if

$$\frac{Y/t - (n/N)}{\sigma_N} > c'(t)$$

where $c'(T) \rightarrow_p z_\alpha \equiv \Phi^{-1}(1 - \alpha)$ since

$$\frac{Y/t - (n/N)}{\sigma_N} \rightarrow_d Z \sim N(0, 1)$$

conditionally on T if $0 < \liminf(n/N) \leq \limsup(n/N) < 1$. Here $\sigma_N^2 = (1 - (t - 1)/(N - 1))(n/N)(1 - n/N)/t$.

(a) Show that

$$\frac{Y/t - (n/N)}{\sigma_N} = \frac{\sqrt{\frac{mn}{N}} \left(\frac{Y}{n} - \frac{X}{m} \right)}{\sqrt{\frac{t}{N} \left(1 - \frac{t-1}{N-1} \right)}}$$

(b) Use the result of (a) to show that

$$\frac{Y/T - (n/N)}{\sigma_N} \rightarrow_d Z \sim N(0, 1)$$

unconditionally under $p_1 = p_2 \equiv p$ (even if $a = \liminf(n/N) < \limsup(n/N) = b$ with possibly $a = 0$ or $b = 1$). [Hint: prove it first under the assumption that $n/N \rightarrow \lambda \in [0, 1]$, and then show that the limit is the same even if n/N does not converge by considering subsequences.]

(c) What is the (unconditional) limiting behavior of the test statistic $(Y/t - (n/N))/\sigma_N$ under local alternatives of the form $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$ assuming that $n/N \rightarrow \lambda \in (0, 1)$?

(d) What does the result of (c) imply about the limiting power of the test under these alternatives?

2. Read TPE (Lehmann and Casella) pages 160 - 162 concerning the notion of *equivariance* of an estimator $\delta = \delta(X)$ under a group of transformations G . Relate this to *invariance* of a (test) function ϕ under a group of transformations G . Illustrate equivariance with two examples.

3. Let X and Y be independent exponential random variables with parameters λ and μ respectively: thus $P(X > x) = \exp(-\lambda x)$ and $P(Y > y) = \exp(-\mu y)$ for $x, y \geq 0$. Let $\theta \equiv \lambda/\mu$.
 - (a) Show that the problem of testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$ is invariant under the group G of transformations $g_c(x, y) = (cx, cy)$, $c > 0$.
 - (b) Find a maximal invariant.
 - (c) Find a UMP invariant test of size α .
 - (d) Show that the problem of testing $H'_0 : \theta = 1$ versus $H'_1 : \theta \neq 1$ is invariant *in addition* under the transformation $g(x, y) = (y, x)$.
 - (e) Find a UMP invariant test of size α .
 - (f) Find UMP invariant tests of the hypotheses in (a) and (d) when X_1, \dots, X_m are i.i.d. Exponential(λ) and Y_1, \dots, Y_n are i.i.d. Exponential(μ).