Statistics 583, Midterm Exam

Wellner; 5/4/2009

Instructions: This is an "in class" and "closed-book" exam. Please do it completely on your own with no books or notes.

1. (30 points) **Define** *any three* of the following terms.
   (a) A maximal invariant with respect to a group G of transformations g on the sample space $\mathcal{X}$).
   (b) A *$G-$invariant test function* ϕ (with respect to a group G.
   (c) The Lévy metric d_L on the set of distribution functions $\mathcal{F}$ on $\mathbb{R}$.
   (d) A metric d_* compatible with the empirical distribution function or empirical measure.
   (e) A Fréchet-differentiable functional $T : \mathcal{F} \rightarrow \mathbb{R}$ with respect to a metric d_*.

2. (30 points) Give a complete **statement** of *any two* of the following results:
   (a) Hoeffding's formula for the distribution of ranks under the alternative.
   (b) The Wald- Wolfowitz-Noether-Hájek finite sampling central limit theorem.
   (c) Varadarajan's theorem concerning weak convergence of the empirical measure $\mathbb{P}_n$ when $X_1, \ldots, X_n$ are i.i.d. P on a metric space (M, d).
   (d) A theorem about the existence of a UMP $G-$invariant test in the case that both the $G-$maximal invariant and the $\overline{G}-$maximal invariant are real-valued.

   Do either problem 3 or problem 4.

3. (36 points) Suppose that $X_1, \ldots, X_m$ are i.i.d. exponential(λ) and that $Y_1, \ldots, Y_n$ are i.i.d. exponential(μ). Thus the density of X_1 is $\lambda \exp(-\lambda x) 1_{[0,\infty)}(x)$. Consider testing $H : \lambda \le \mu$ versus $K : \lambda > \mu$.
   (a) Show that this testing problem is invariant with respect to the group of scale changes G given by $g_c(\underline{x}, \underline{y}) = (c\underline{x}, c\underline{y})$ where $c > 0$.
   (b) Find the UMP G-invariant test of H versus K. [Hint you may use the fact that the family of distributions $\delta^{-1} F_{r,s} : \delta > 0$ has monotone likelihood ratio.
   (c) Specify as exactly as possible how you would carry out the test derived in (b).

4. (36 points) Suppose that X_1 has continuous distribution function F and Y_1, Y_2 are independent of X_1 and themselves independent with distribution function $G = F^2$. Let $\underline{Q} = (Q_1, Q_2)$ denote the ordered Y ranks.
   (a) Is $F <_s G$? (Explain why or why not.)
   (b) Compute the probabilities $P_{F,G}(\underline{Q} = \underline{q})$ for $\underline{q} \in \{(1, 2), (1, 3), (2, 3)\}$ under the alternative $G = F^2$.
   (c) Use (b) to find the most powerful rank test of $F = G$ versus $G = F^2$ at level $\alpha = 1/3$.

Do either problem 5 or problem 6.

5. (36 points) (a) State the general linear model in its "canonical form", specifying the parameter space Θ of the general model, and the canonical null hypothesis Θ_0.
(b) Under what group $G = G_1 \oplus G_2 \oplus G_3 \oplus G_4$ of transformations on the sample space are both Θ and Θ_0 invariant under the induced group $\overline{G}$ on the parameter space?
(c) What is the usual form of the general linear model? Explain the relationship of the "usual form" to the "canonical form", including a transformation which relates them explicitly.

6. (36 points) Suppose that F is a distribution function on $(0, \infty)$ given in terms of a distribution function H on $(0, \infty)$ by

$$F(x) = \frac{1}{\mu} \int_0^x y dH(y)$$

where $\mu \equiv \int_0^\infty y dH(y) < \infty$. Then F is the *length-biased distribution* corresponding to H. Suppose also that $\int_0^\infty y^{-1} dH(y) < \infty$
(a) Show that the distribution function H can be expressed in terms of F by

$$H(x) \equiv H_F(x) \equiv \frac{\int_0^x y^{-1} dF(y)}{\int_0^\infty y^{-1} dF(y)}.$$

(b) Fix $x_0 \in (0, \infty)$, and let $T(F) \equiv H_F(x_0)$ where H_F is as in (a). Find the Gateaux derivative of $T(F)$ at F and the influence curve $IC(x; T, F)$.
(c) Use the calculation of (b) to "guess" the asymptotic variance of the limiting distribution of $\sqrt{n}(T(\mathbb{F}_n) - T(F))$.
(d) How would you proceed to prove the result suggested by (b) and (c)?