Statistics 583, Problem Set 2 Solutions

Wellner; 4/15/2009

1. (Problem 10, page 249, Ferguson, MS) Let $\Theta = \{(\Delta, \pi_1, \ldots, \pi_n) : \ \Delta \ge 0, \ \pi = (\pi_1, \ldots, \pi_n)$ is a permutation of $\{1, \ldots, n\}\}$, and let the distribution of $X_1, \ldots, X_n$ given $\theta = (\Delta, \pi_1, \ldots, \pi_n)$ be as independent random variables with gamma distributions, $X_i \sim \text{Gamma}(\alpha, \beta^{-1}\exp(-\Delta b_{\pi_i}))$ where $\alpha > 0$, $\beta > 0$, and $b_1, \ldots, b_n$ are known real numbers with $\sum_1^n b_i > 0$. Consider testing the hypothesis $H : \Delta = 0$ versus the alternative $K : \Delta > 0$. (This is a Gamma-regression model with covariates or predictors b_i in which the relationship between the responses X_i and the covariates b_i have become scrambled or mixed up: we unfortunately don't know the right pairing of X_i and b_i, but we do know that some permutation of the b_i's is correct. Note that problem 11 in Ferguson, MS, gives a more realistic version of the problem in which β is unknown.)
(a) Show that this problem is invariant under the group of permutations of $(X_1, \ldots, X_n)$, and that the distribution of the maximal invariant $(Y_1, \ldots, Y_n) \equiv (X_{(1)}, \ldots, X_{(n)})$(the order statistics) has density

$$f_{\underline{Y}}(\underline{y}|\Delta) = \frac{(\prod_1^n y_i)^{\alpha-1}\exp(-\alpha\Delta\sum_1^n b_i)}{\Gamma(\alpha)^n\beta^{n\alpha}} \sum_{\pi^*\in\Pi} \exp\left\{-\frac{1}{\beta}\sum_{i=1}^n y_i \exp(-\Delta b_{\pi^*(i)})\right\}$$

for $y_1 < \cdots < y_n$ and zero elsewhere where $\sum_{\pi^*\in\Pi}$ denotes the sum over all permutations π of $\{1, \ldots, n\}$. (b) Show that the locally best invariant test of H versus K (i.e. the test which maximizes the slope of the power function at the null hypothesis) is to reject H when $\sum_{i=1}^n X_i$ is too large.

Solution: Let G be the permutation group

$$G = \{g : \ g(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)}), \ \pi \in \Pi\}.$$

Then, if $\underline{X} \sim P_\theta$, for $g = g_{\pi'} \in G$, $g(\underline{X}) \sim P_{\overline{g}(\theta)}$ with $\overline{g}(\theta) = (\Delta, \pi \circ \pi') = (\Delta, (\pi_{\pi'(1)}, \ldots, \pi_{\pi'(n)}))$. Thus the hypotheses are invariant under G. The order statistics are a $G-$MI. But of course the X_i's are *not* identically distributed. The joint density of the X_i's is given by

$$\begin{aligned} f(\underline{x}; \Delta, \pi) &= \prod_{i=1}^n \frac{x_i^{\alpha-1}e^{-\alpha\Delta b_{\pi(i)}}}{\Gamma(\alpha)\beta^\alpha}\exp(-\beta^{-1}e^{-\Delta b_{\pi(i)}}x_i) \\ &= \frac{\exp(-\Delta\sum_{j=1}^n b_j)(\prod_{i=1}^n x_i)^{\alpha-1}}{\Gamma(\alpha)^n\beta^{n\alpha}} \exp\left(-\beta^{-1}\sum_{i=1}^n x_i e^{-\Delta b_{\pi(i)}}\right). \end{aligned}$$

If $X_{(\cdot)} = (X_{(1)}, \ldots, X_{(n)})$ denotes the order statistics, then by problem 3(c) of problem set #9, Statistics 582,

$$
\begin{aligned}
& f_{X_{(\cdot)}}(\underline{x}_{(\cdot)}; \Delta, \pi) \\
& \quad = \sum_{\pi' \in \Pi} f(\pi' \underline{x}; \Delta, \pi) \\
& \quad = \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \exp(-\alpha\Delta \sum b_j) \sum_{\pi' \in \Pi} \prod_{i=1}^{n} x_{\pi'(i)}^{\alpha-1} \exp\left\{ -\frac{1}{\beta} \sum_{i=1}^{n} x_{(\pi'(i))} \exp(-\Delta b_{\pi_i}) \right\} \\
& \quad = \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \exp(-\alpha\Delta \sum b_j) \prod_{i=1}^{n} x_{(i)}^{\alpha-1} \sum_{\pi^* \in \Pi} \exp\left\{ -\frac{1}{\beta} \sum_{i=1}^{n} x_{(i)} \exp(-\Delta b_{\pi_i^*}) \right\}
\end{aligned}
$$

with $\pi^* \equiv (\pi')^{-1} \circ \pi$. Note that this distribution depends only on the $\overline{G}-$MI Δ (and not on π).
(b) Now

$$
\begin{aligned}
l(\Delta | \underline{X}_{(\cdot)}) & \equiv \log f_{X_{(\cdot)}}(\underline{X}(\cdot); \Delta) \\
& = \log \left\{ \sum_{\pi^* \in \Pi} \exp\left\{ -\frac{1}{\beta} \sum_{i=1}^{n} X_{(i)} \exp(-\Delta b_{\pi^*(i)}) \right\} \right\} \\
& \qquad - \alpha\Delta \sum_{i=1}^{n} b_i + \text{constant in} \quad \Delta
\end{aligned}
$$

so that

$$
\begin{aligned}
\dot{\mathrm{l}}_{\Delta}(\underline{X}_{(\cdot)}; \Delta = 0) & = \frac{1}{\sum_{\pi^*} \exp(\cdots)} \Big|_{\Delta=0} \\
& \qquad \cdot \sum_{\pi^* \in \Pi} \exp(\cdots) \Big|_{\Delta=0} \left\{ \frac{1}{\beta} \sum_{i=1}^{n} X_{(i)} \exp(-\Delta b_{\pi^*(i)})(-b_{\pi^*(i)}) \Big|_{\Delta=0} \right\} \\
& \qquad\quad -\alpha \sum_{j=1}^{n} b_j \\
& = \frac{1}{\beta n!} \sum_{\pi^*} \left\{ \sum_{i=1}^{n} X_{(i)} b_{\pi^*(i)} \right\} - \alpha \sum_{i=1}^{n} b_i \\
& = \frac{1}{\beta n!} \sum_{i=1}^{n} X_{(i)} \sum_{\pi^*} b_{\pi^*(i)} - \alpha \sum_{i=1}^{n} b_i \\
& = \frac{1}{\beta} \overline{b} \sum_{i=1}^{n} X_{(i)} - \alpha n \overline{b} = \frac{n\overline{b}}{\beta} (\overline{X} - \alpha\beta),
\end{aligned}
$$

and hence the locally MP invariant test rejects for large values of $\sum_{i=1}^n X_i$.

For an extension to unknown β and to testing $H_0 : \Delta = 0$ versus $K : \Delta \neq 0$, see Ferguson, problem 11, page 249. In this version of the problem $\sum_1^n b_i = 0$ *is* assumed.

2. In class in the context of Example 6.3.14 we developed a UMP invariant test under normal hypotheses: "reject H if $T \equiv \sqrt{n}\overline{X}/S > t_{n-1,\alpha}(\delta_0)$ where $\delta_0 \sqrt{n}\Phi^{-1}(1 - p_0)$.
   (a) Study the limiting power of this test assuming that the Y's (and hence also the X's) have $E(Y^2) < \infty$ and that the Y's are i.i.d. according to the location-scale family $F_{\mu,\sigma}(x) = F_0((x - \mu)/\sigma)$. (You will need to decide on how to specify "local alternatives".)
   (b) Now consider the alternative test based on the empirical d.f. of the Y's: "reject $H : p \geq p_0$ (in favor of $K : \ p < p_0$) if $n\mathbb{F}_n(y_0) \leq c_{n,\alpha}$ where $c_{n,\alpha}$ is the largest integer satisfying $P(\text{Bin}(n, p_0) \leq c_{n,\alpha}) < \alpha$. Study the limiting power of this test assuming local alternatives of the form $p_n = p_0 - c/\sqrt{n}$ with $c > 0$.
   (c) Compare the asymptotic power of the tests in (a) and (b) assuming that F_0 is normal. [Hint: it might be helpful to read example 6.4.2 on page 33 of section 6.4.]

   Solution: (a) As discussed in class on 4/7, the first step is to understand the limiting behavior of the critical points under normal theory. We know that under X_i's i.i.d. $N(\theta_0, \sigma_0^2)$ it follows that

$$\begin{pmatrix} \sqrt{n}(\overline{X} - \theta_0)/\sigma_0 \\ \sqrt{n}(S_n^2/\sigma_0^2 - 1) \end{pmatrix} \to_d N_2\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right).$$

   Since under normality we have, with $c_0 \equiv \theta_0/\sigma_0 = \Phi^{-1}(1 - p_0)$,

$$\begin{aligned} \alpha &= P_{\theta_0,\sigma_0}\left(\frac{\sqrt{n}\overline{X}}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right) \\ &= P_{\theta_0,\sigma_0}\left(\frac{\sqrt{n}(\overline{X} - \theta_0)}{S_n} + \frac{\sqrt{n}\theta_0}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right) \quad (1) \\ &= P_{\theta_0,\sigma}\left(\frac{\sqrt{n}(\overline{X} - \theta_0)}{S_n} + \frac{\theta_0}{\sigma_0}\sqrt{n}\left(\frac{\sigma_0}{S_n} - 1\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}c_0\right). \quad (2) \end{aligned}$$

   Now by the delta-method,

$$\begin{pmatrix} \sqrt{n}(\overline{X} - \theta_0)/\sigma_0 \\ \sqrt{n}(\sigma_0/S_n - 1) \end{pmatrix} \to_d \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N_2\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\right).$$

and hence

$$\frac{\sqrt{n}(\overline{X}-\theta_0)}{S_n}+\frac{\theta_0}{\sigma_0}\sqrt{n}\left(\frac{\sigma_0}{S_n}-1\right)\to_d Z_1+c_0Z_2\sim N(0,1+(1/2)c_0^2).$$

Letting $\gamma\equiv\lim_{n\to\infty}\{t_{n-1,\alpha}(\sqrt{n}c_0)-\sqrt{n}c_0\}$, and taking limits across the identity in (2) yields

$$\alpha=P(Z_1+c_0^2Z_2>\gamma)=P(\sqrt{1+(1/2)c_0^2}Z>\gamma)$$

where $Z\sim N(0,1)$, and this forces $\gamma=\sqrt{1+(1/2)c_0^2}z_\alpha$ with $z_\alpha=\Phi^{-1}(1-\alpha)$.

Now we are ready to treat the behavior of the size and power of this test when $F_{\mu,\sigma}(x)=F_0((x-\mu)/\sigma)$ and F_0 is a distribution with mean 0, variance 1, and finite fourth moment. This implies that the X_i's are i.i.d. with mean θ and variance σ^2. From problem 3, problem set #3, Stat 581, we know that

$$\sqrt{n}\begin{pmatrix}\overline{X}_n-\theta\\ S_n^2-\sigma^2\end{pmatrix}\to_d \underline{Z}\sim N_2(0,\Sigma)$$

where

$$\begin{pmatrix}\sigma^2 & \mu_3\\ \mu_3 & \mu_4-\sigma^4\end{pmatrix}.$$

Furthermore, from the solution of part (b) of the same problem we know that with $g(u,v)=u/\sqrt{v}$

$$\begin{aligned}\sqrt{n}(\overline{X}_n/S_n-\theta/\sigma) &= \sqrt{n}(g(\overline{X}_n,S_n^2)-g(\theta,\sigma^2))\\ &\to_d \nabla g\cdot\underline{Z}\sim N(0,\nabla g^T\Sigma\nabla g)\equiv N(0,V^2)\end{aligned}$$

and it is easy to calculate that

$$\begin{aligned}V^2 &= \nabla g^T\Sigma\nabla g\\ &= \frac{1}{\sigma^4}\left\{\sigma^4-\theta\mu_3+\frac{1}{4}c^2(\mu_4-\sigma^4)\right\}\\ &= 1-c\gamma_1+\frac{1}{4}c^2(2+\gamma_2)\equiv V^2(c,\gamma_1,\gamma_2)\equiv V^2\end{aligned}$$

where $c\equiv\theta/\sigma$, $\gamma_1\equiv\mu_3/\sigma^3$, and $\gamma_2\equiv\mu_4/\sigma^4-3$. Note that when the X_i's are normal (so $\gamma_1=\gamma_2=0$), this reduces to $1+c^2/2$. Thus the size of our normal theory test when the Y_i's are really i.i.d. F_{μ_0,σ_0} (with corresponding parameters

θ_0 and σ_0 for the X_i's) is

$$\begin{aligned}
&P_{\theta_0,\sigma_0}\left(\frac{\sqrt{n}\overline{X}}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right)\\
&= P_{\theta_0,\sigma_0}\left(\sqrt{n}\left(\frac{\overline{X}}{S_n} - \frac{\theta_0}{\sigma_0}\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}c_0\right)\\
&\to P(V_0 Z > \sqrt{1+2^{-1}c_0^2}z_\alpha) = P(Z > V_0^{-1}\sqrt{1+2^{-1}c_0^2}z_\alpha)\\
&= 1 - \Phi(V_0^{-1}\sqrt{1+2^{-1}c_0^2}z_\alpha)
\end{aligned}$$

where $V_0^2 \equiv V^2(c_0, \gamma_1(F_0), \gamma_2(F_0))$. When $F_0 = \Phi$ then $\gamma_1(\Phi) = \gamma_2(\Phi) = 0$, $V_0\sqrt{1+2^{-1}c_0^2} = 1$, and the asymptotic size of the test is α (as it must be). However for distributions F_0 with $\gamma_1(F_0) \neq 0$ or $\gamma_2(F_0) \neq 0$, the asymptotic size of this test is generally not α.

How does the power of the normal theory test behave for local alternatives of the form $p_n = p_0 - d/\sqrt{n}$ for $c > 0$? In this case $c_n \equiv \theta_n/\sigma_n = \Phi^{-1}(1-p_n)$ where $\theta_n \to \theta_0$ and $\sigma_n \to \sigma_0$, and we have

$$\begin{aligned}
\sqrt{n}(c_n - c_0) &= \sqrt{n}(\Phi^{-1}(1-p_n) - \Phi^{-1}(1-p_0))\\
&= \frac{\Phi^{-1}(1-p_n) - \Phi^{-1}(1-p_0)}{p_n - p_0} \cdot \sqrt{n}(p_n - p_0)\\
&= \frac{d}{du}\Phi^{-1}(1-u)\Big|_{u=p_0} \cdot (-d)\\
&= \frac{-1}{\phi(\Phi^{-1}(1-p_0))}(-d) = \frac{d}{\phi(\Phi^{-1}(1-p_0))}.
\end{aligned}$$

Thus the power of this normal theory test under local alternatives of this form is

$$\begin{aligned}
\beta_{\text{Norm}}(\theta_n, \sigma_n) &= P_{\theta_n,\sigma_n}\left(\frac{\sqrt{n}\overline{X}}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right)\\
&= P_{\theta_n,\sigma_n}\left(\sqrt{n}\left(\frac{\overline{X}}{S_n} - \frac{\theta_n}{\sigma_n}\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}\frac{\theta_n}{\sigma_n}\right)\\
&= P_{\theta_n,\sigma_n}\left(\sqrt{n}\left(\frac{\overline{X}}{S_n} - \frac{\theta_n}{\sigma_n}\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}c_0 - \sqrt{n}(c_n - c_0)\right)\\
&\to P(V_0 Z > (1+2^{-1}c_0^2)^{1/2}z_\alpha - d/\phi(\Phi^{-1}(1-p_0))).
\end{aligned}$$

When $F_0 = \Phi$, $V_0^2 = (1+2^{-1}c_0^2)$, and the limiting power becomes, under normality,

$$P\left(Z > z_\alpha - \frac{d}{(1+2^{-1}c_0^2)^{1/2}\phi(\Phi^{-1}(1-p_0))}\right).$$

In the terminology used in section 6.4, the normal theory test has *efficacy*

$$\epsilon_{Norm} = \frac{1}{(1+2^{-1}c_0^2)^{1/2}\phi(\Phi^{-1}(1-p_0))} = \frac{1}{(1+2^{-1}\Phi^{-1}(1-p_0)^2)^{1/2}\phi(\Phi^{-1}(1-p_0))}$$

under normality.

(b) If we use the alternative test "reject $H: p \ge p_0$ when $n\mathbb{F}_n(y_0) \le c_{n,\alpha}$ where $P(\text{Bin}(n,p_0) \le c_{n,\alpha}) < \alpha$ (and $c_{n,\alpha}$ is the largest such integer), then since

$$\frac{n\mathbb{F}_n(y_0) - np_0}{\sqrt{np_0(1-p_0)}} \to_d Z \sim N(0,1)$$

we deduce that $(c_{n,\alpha} - np_0)/\sqrt{np_0(1-p_0)} \to z_\alpha \equiv \Phi^{-1}(\alpha)$. Thus the power of the binomial test under alternatives of the form $p_n = p_0 - c/\sqrt{n}$ is

$$\begin{aligned}
\beta_{\text{Bin}}(p_n) &= P_{p_n}(n\mathbb{F}_n(y_0) \le c_{n,\alpha}) \\
&= P_{p_n}\left(\frac{n\mathbb{F}_n(y_0) - np_n}{\sqrt{np_n(1-p_n)}} \le \frac{c_{n,\alpha} - np_n}{\sqrt{np_n(1-p_n)}}\right) \\
&= P_{p_n}\left(\frac{n\mathbb{F}_n(y_0) - np_n}{\sqrt{np_n(1-p_n)}} \le \frac{c_{n,\alpha} - np_0}{\sqrt{np_0(1-p_0)}}\frac{\sqrt{np_0(1-p_0)}}{\sqrt{np_n(1-p_n)}} + \frac{\sqrt{n}c}{\sqrt{np_n(1-p_n)}}\right) \\
&\to P\left(Z \le z_\alpha + \frac{c}{\sqrt{p_0(1-p_0)}}\right).
\end{aligned}$$

Thus the *efficacy* of the binomial test is

$$\epsilon_{Bin} = \frac{1}{\sqrt{p_0(1-p_0)}}.$$

(c) As discussed in section 6.4, one convenient way to summarize the comparison between the two tests in (a) and (b) is via their *Pitman efficiency*, which as noted on page 35 is the ratio of the squares of the efficacies. Here we get

$$e_{Bin,Norm} = \left\{\frac{\frac{1}{\sqrt{p_0(1-p_0)}}}{\frac{1}{(1+2^{-1}\Phi^{-1}(1-p_0)^2)^{1/2}\phi(\Phi^{-1}(1-p_0))}}\right\}^2 = \frac{(1+\Phi^{-1}(1-p_0)^2/2)\phi^2(\Phi^{-1}(1-p_0))}{p_0(1-p_0)}.$$

Here is a plot of this relative efficiency as a function of p_0. It is fairly flat, with values in the range of .63 to about .65 over the interval $.15 \le p_0 \le 1 - p_0$, but declines rapidly to 0 for $p_0 \le .15$ or $p_0 \ge .85$. Note that the value at $p_0 = .5$ is $2/\pi$, which is the Pitman efficiency of the sign test relative to the $t-$test at normality.

Of course the normal theory test does not have asymptotically correct size when the assumption of normality is violated, while the binomial test maintains the correct size asymptotically regardless of the underlying distribution of the Y's (and hence of the X's).

3. Suppose that X_{ijk}, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ satisfy the general linear model with $\xi_{ijk} = \xi + \mu_i + \eta_j + \delta_{ij}$ where $\sum_i \mu_i = 0$, $\sum_j \eta_j = 0$, $\sum_j \delta_{ij} = 0$ for all i, and $\sum_i \delta_{ij} = 0$ for all j. (δ_{ij} is called the interaction effect of the ith row and the jth column.)
(a) Show that

$$\begin{aligned} S^2 &= \sum\sum\sum(X_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \\ &= \sum\sum\sum(X_{ijk} - \overline{X}_{ij\cdot})^2 \\ &\quad + \sum\sum\sum(\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots} - \delta_{ij})^2 \\ &\quad + \sum\sum\sum(\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots} - \mu_i)^2 + \sum\sum\sum(\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdots} - \eta_j)^2 \\ &\quad + \sum\sum\sum(\overline{X}_{\cdots} - \xi)^2 \end{aligned}$$

where $\overline{X}_{ij\cdot} = \sum_k X_{ijk}/K$, and so on.
(b) Find the UMP invariant test of the hypothesis of no row effect $H_0 : \mu_1 = \ldots = \mu_I = 0$. What is the distribution of the test statistic under the general linear hypothesis – including the noncentrality parameter?
(c) Find the UMP invariant test of the hypothesis of no interaction effect $H_0 : \delta_{ij} = 0$ for all i, j. What is the distribution of the test statistic under the general linear hypothesis?

Solution: (a) We write $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} \equiv \sum$, and note that

$$\begin{aligned}
S^(\xi) &= \sum (X_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \\
&= \sum (X_{ijk} - \overline{X}_{ij\cdot} + \overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots} \\
&\qquad + \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots} + \overline{X}_{\cdot j\cdot} - \overline{X}_{\cdots} \\
&\qquad + \overline{X}_{\cdots} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \\
&= \sum (X_{ijk} - \overline{X}_{ij\cdot})^2 \\
&\quad + \sum (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots} - \delta_{ij})^2 \\
&\quad + \sum (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots} - \mu_i)^2 + \sum (\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdots} - \eta_j)^2 \\
&\quad + \sum (\overline{X}_{\cdots} - \xi)^2
\end{aligned}$$

where the ten cross product terms all vanish because

$$\begin{aligned}
&\sum_i (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots}) = 0, \quad \text{for all } j, \\
&\sum_j (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots}) = 0, \quad \text{for all } i, \\
&\sum_i (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots}) = 0, \quad \text{for all } j, \\
&\sum_j (\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdots}) = 0, \quad \text{for all } i, \\
&\sum_k (\overline{X}_{ijk} - \overline{X}_{ij\cdot}) = 0, \quad \text{for all } i, j.
\end{aligned}$$

(b) It follows from (a) that the least squares estimators of ξ_{ijk} are given by $\widehat{\xi}_{i,j,k} = \widehat{\xi} + \widehat{\mu}_i + \widehat{\eta}_j + \widehat{\delta}_{ij}$ where

$$\begin{aligned}
\widehat{\xi} &= \overline{X}_{\cdots}, \\
\widehat{\mu}_i &= \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots}, \\
\widehat{\eta}_j &= \overline{X}_{\cdot j\cdot} - \overline{X}_{\cdots}, \\
\widehat{\delta}_{ij} &= \overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots}.
\end{aligned}$$

Thus the residual sum of squares under the big model is $\sum (X_{ijk} - \overline{X}_{ij\cdot})^2$. On the other hand, under the hypothesis of no row effect, the least squares estimators $\hat{\hat{\xi}}_{ijk} = \hat{\hat{\xi}} + \hat{\hat{\mu}}_i + \hat{\hat{\eta}}_j + \hat{\hat{\delta}}_{ij}$ of ξ_{ijk} are now given by $\hat{\hat{\xi}} = \widehat{\xi}$, $\hat{\hat{\eta}}_j = \widehat{\eta}_j$, and $\hat{\hat{\delta}}_{ij} = \widehat{\delta}_{ij}$, but $\hat{\hat{\mu}}_i = 0$, and then the residual sum of squares is

$$S^2(\hat{\hat{\xi}}) = \sum (X_{ijk} - \overline{X}_{ij\cdot})^2 + \sum (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots})^2.$$

Here $n = IJK$, $k = IJ$, and $r = I - 1$. Thus the F statistic for testing $H:\ \mu_1 = \cdots = \mu_i = 0$ is given by

$$F = \frac{\sum(\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots})^2/(I-1)}{\sum(X_{ijk} - \overline{X}_{ij\cdot})^2/(IJK - IJ)}.$$

which has an $F_{I-1,IJ(K-1)}(\delta^2)$ distribution where $\delta^2 = JK\sum_{i=1}^{I}\mu_i^2/\sigma^2$.

(c) The least squares estimators of ξ_{ijk} under the big model are exactly the same as in (b): $\widehat{\xi}_{i,j,k} = \widehat{\xi} + \widehat{\mu}_i + \widehat{\eta}_j + \widehat{\delta}_{ij}$, and the residual sum of squares under the big model is also exactly the same. Under the hypothesis H that all $\delta_{ij} = 0$, the least squares estimators $\hat{\hat{\xi}}_{ijk} = \hat{\hat{\xi}} + \hat{\hat{\mu}}_i + \hat{\hat{\eta}}_j + \hat{\hat{\delta}}_{ij}$ of ξ_{ijk} are now given by $\hat{\hat{\xi}} = \widehat{\xi}$, $\hat{\hat{\eta}}_j = \widehat{\eta}_j$, and $\hat{\hat{\mu}}_i = \widehat{\mu}_i$, but now $\hat{\hat{\delta}}_{ij} = 0$, and the residual sum of squares is

$$S^2(\hat{\hat{\xi}}) = \sum(X_{ijk} - \overline{X}_{ij\cdot})^2 + \sum(\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots})^2.$$

4. Consider a two-way classification X_{ij}, $i = 1, \ldots, I$, $j = 1, \ldots, J$ with the assumptions of the general linear hypothesis for which $EX_{ij} = \alpha + \beta z_i + \eta_j$, where α, β, and η_j are unknown parameters subject to the restriction $\sum\eta_j = 0$, and where z_i are known numbers for which $\sum_1^I z_i = 0$ and $\sum_1^I z_i^2 = 1$.
(a) Find the UMP invariant test of the hypothesis

$$H_0:\ \eta_1 = \eta_2 = \cdots = \eta_J = 0.$$

(b) What is the distribution of the test statistic under the general linear hypothesis?

Solution: Here $\xi = (\xi_{ij},\ i = 1, \ldots, I,\ j = 1, \ldots, J)$ with $\xi_{ij} = \alpha + \beta z_i + \eta_j$ where $\sum_i z_i = 0$, $\sum_1^I z_i^2 = 1$, and $\sum_j \eta_j = 0$. Thus the ξ_{ij}'s are in a $J+1$-dimensional subspace of R^n where $n = IJ$. The null hypothesis imposes $r = J-1$ restrictions, and under the null hypothesis the ξ_{ij}'s are in a $2-$dimensional subspace of R^n. It is straightforward to see that the unrestricted least squares estimators are given by

$$\widehat{\alpha} = \overline{X}_{\cdot\cdot} = \frac{1}{IJ}\sum_{ij} X_{ij},$$

$$\widehat{\beta} = \sum_{i=1}^{I}\overline{X}_{i\cdot}z_i, \qquad \text{where } \overline{X}_{i\cdot} = \frac{1}{J}\sum_j X_{ij},$$

$$\widehat{\eta}_j = \overline{X}_{\cdot j} - \overline{X}_{\cdots}.$$

Under the null hypothesis that $\eta_1 = \eta_2 = \cdots = \eta_J = 0$,

$$\hat{\hat{\alpha}} = \widehat{\alpha} = \overline{X}_{..},$$
$$\hat{\hat{\beta}} = \widehat{\beta} = \sum_{i=1}^{I} \overline{X}_{i.} z_i,$$

as before, but now $\hat{\hat{\eta}}_j = 0$, $j = 1, \dots, J$. Hence

$$\widehat{\xi}_{ij} - \hat{\hat{\xi}}_{ij} = \widehat{\eta}_j = \overline{X}_{\cdot j} - \overline{X}_{..}$$

and

$$\sum_{i,j} (\widehat{\xi}_{ij} - \hat{\hat{\xi}}_{ij})^2 = \sum (\overline{X}_{\cdot j} - \overline{X}_{..})^2 = I \sum_{j=1}^{J} (\overline{X}_{\cdot j} - \overline{X}_{..})^2$$

while

$$\sum_{i,j} (X_{ij} - \widehat{\xi}_{ij})^2 = \sum_{i,j} (X_{ij} - \overline{X}_{\cdot j} - \widehat{\beta} z_i)^2.$$

Thus the F statistic becomes

$$F = \frac{I \sum_{j=1}^{J} (\overline{X}_{\cdot j} - \overline{X}_{..})^2 / (J-1)}{\sum_{i,j} (X_{ij} - \overline{X}_{\cdot j} - \widehat{\beta} z_i)^2 / (IJ - J - 1)}.$$

Under the general linear hypothesis the noncentrality parameter is

$$\delta^2 = \frac{1}{\sigma^2} \sum_{i,j} \widehat{\xi}_{ij}(\underline{\xi}) - \hat{\hat{\xi}}(\underline{\xi}))^2 = \frac{I \sum_{j=1}^{J} \eta_j^2}{\sigma^2}.$$

Hence under the general linear hypothesis

$$F \sim F_{J-1, IJ-J-1}(\delta^2).$$

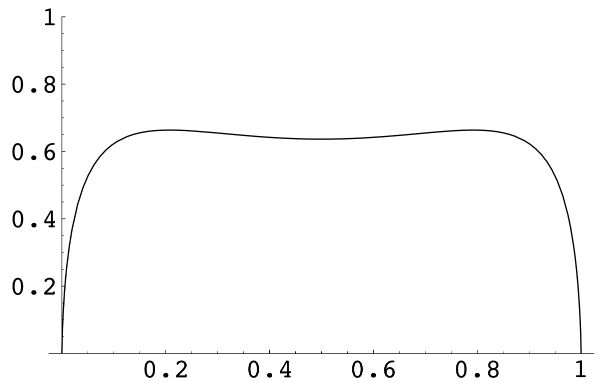

Figure 1: Pitman ARE plot as a function of p_0