Statistics 583, Problem Set 1 Solutions

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1. (Continuation of problem 4(f), Stat 582 final exam.) Suppose that $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$ are independent. In problem 4 of the 582 final exam we derived the UMP unbiased (conditional) test of level α for testing $H : p_2 \le p_1$ versus $K : p_2 > p_1$. It involves rejecting H if $Y > c(t)$ relative to the conditional (hypergeometric) distribution of Y conditional on $T = X + Y = t$, or equivalently if

$$\frac{Y/t - (n/N)}{\sigma_N} > c'(t)$$

where $c'(T) \to_p z_\alpha \equiv \Phi^{-1}(1-\alpha)$ since

$$\frac{Y/t - (n/N)}{\sigma_N} \to_d Z \sim N(0,1)$$

conditionally on T if $0 < \liminf(n/N) \le \limsup(n/N) < 1$. Here $\sigma_N^2 = (1 - (t-1)/(N-1))(n/N)(1-n/N)/t$.
(a) Show that

$$\frac{Y/t - (n/N)}{\sigma_N} = \frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n} - \frac{X}{m}\right)}{\sqrt{\frac{t}{N}\left(1 - \frac{t-1}{N-1}\right)}}.$$

(b) Use the result of (a) to show that

$$\frac{Y/T - (n/N)}{\sigma_N} \to_d Z \sim N(0,1)$$

unconditionally under $p_1 = p_2 \equiv p$ (even if $a = \liminf(n/N) < \limsup(n/N) = b$ with possibly $a = 0$ or $b = 1$). [Hint: prove it first under the assumption that $n/N \to \lambda \in [0,1]$, and then show that the limit is the same even if n/N does not converge by considering subsequences.]
(c) What is the (unconditional) limiting behavior of the test statistic $(Y/t - (n/N))/\sigma_N$ under local alternatives of the form $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$ assuming that $n/N \to \lambda \in (0,1)$?
(d) What does the result of (c) imply about the limiting power of the test under these alternatives?

Solution: (a) Note that

$$\begin{aligned}\frac{Y}{T}-\frac{n}{N} &= \frac{1}{T}\left(Y-\frac{n}{N}T\right)=\frac{1}{T}\left(Y-\frac{n}{N}(X+Y)\right)\\ &= \frac{1}{T}\left(\frac{m}{N}Y-\frac{n}{N}X\right)=\frac{mn}{NT}\left(\frac{Y}{n}-\frac{X}{m}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{Y/t-(n/N)}{\sigma_N} &= \frac{Y/t-n/N}{\sqrt{(1-\frac{n-1}{N-1})\frac{n}{N}\frac{m}{N}\frac{1}{t}}}=\frac{\sqrt{t}(Y/t-n/N)}{\sqrt{(1-\frac{n-1}{N-1})\frac{n}{N}\frac{m}{N}}}\\ &= \frac{\frac{mn}{N}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{T(1-\frac{T-1}{N-1})\frac{mn}{N^2}}}=\frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}}\end{aligned}$$

as claimed.
(b) Now $(\sqrt{m}(X/m-p),\sqrt{n}(Y/n-p))\to_d(Z_1,Z_2)$ where $Z_j\sim N(0,p(1-p))$, $j=1,2$ are independent. Thus from (a), under the assumption that $n/N\to\lambda\in[0,1]$,

$$\begin{aligned}V_N\equiv\frac{Y/T-(n/N)}{\sigma_N} &= \frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}}\\ &= \frac{\sqrt{\frac{m}{N}}\sqrt{n}(Y/n-p)-\sqrt{\frac{n}{N}}\sqrt{m}(X/m-p)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}}\\ &\to_d \frac{\sqrt{1-\lambda}Z_2-\sqrt{\lambda}Z_1}{\sqrt{p(1-p)}}\sim N(0,1)\end{aligned}$$

where we used $T/N=(m/N)(X/m)+(n/N)(Y/n)\to_p(1-\lambda)p+\lambda p=p$.

If $\lambda_N\equiv n/N\not\to$, then since $\lambda_N\in[0,1]$, for any initial subsequence $\{\lambda_{N'}\}$, there exists a further convergent subsequence $\{\lambda_{N''}\}$; i.e. $\lambda_{N''}\to$ some $\lambda\in[0,1]$. By the same argument as above, for this subsequence $V_{N''}\to_d Z\sim N(0,1)$. Since the limiting distribution is the same for any such initial subsequence $\{V_{N'}\}$, we conclude that the full sequence $\{V_N\}$ satisfies $V_N\to_d Z\sim N(0,1)$ under $p_1=p_2=p$. (This argument is completely analogous to the following fact concerning real numbers: a sequence $\{x_n\}$ of real numbers satisfies $x_n\to x$ if and only if each subsequence $\{x_{n'}\}$ contains a further subsequence $\{x_{n''}\}$ such that $x_{n''}\to x$. See Billingsley (1968), *Convergence of Probability Measures*, theorem 2.3, page 16.)

(c) Under local alternatives $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$, we have

$$\begin{aligned}\sqrt{n}(Y/n - p_1) &= \sqrt{n}(Y/n - p_1 - c/\sqrt{N}) + \sqrt{n}c/\sqrt{N} \\ &= \sqrt{n}(Y/n - p_{2,N}) + c\sqrt{\frac{n}{N}} \\ &\to_d Z_2 + c\sqrt{\lambda} \sim N(c\sqrt{\lambda}, p_1(1-p_1))\end{aligned}$$

under the assumption that $\lambda_N = n/N \to \lambda$. Then it follows that $(\sqrt{m}(X/m - p_1), (\sqrt{n}(Y/n - p_1)) \to_d (Z_1, Z_2 + c\sqrt{\lambda})$ and hence

$$V_N \to_d \frac{\sqrt{1-\lambda}(Z_2 + c\sqrt{\lambda}) - \sqrt{\lambda}Z_1}{\sqrt{p_1(1-p_1)}} \sim N\left(c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}}\ ,\ 1\right).$$

(d) The limiting distribution under local alternatives found in (c) implies that the power of the test based on V_N statisfies

$$\begin{aligned}\lim_{N\to\infty} \beta((p_1, p_{2,N})) &= \lim_{N\to\infty} P_{(p_1,p_{2,N})}(V_N > z_\alpha) \\ &= P\left(Z + c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}} > z_\alpha\right) \\ &= 1 - \Phi\left(z_\alpha - c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}}\right).\end{aligned}$$

2. Read TPE (Lehmann and Casella) pages 160 - 162 concerning the notion of *equivariance* of an estimator $\delta = \delta(X)$ under a group of transformations G. Relate this to *invariance* of a (test) function ϕ under a group of transformations G. Illustrate equivariance with two examples.

Solution: Let $X \sim P_\theta$ for $\theta \in \Theta$, and suppose that we want to estimate some function $h(\theta) \in \mathcal{H}$. For a group of transformations G on the sample space $\mathcal{X}$, we typically induce a group $\overline{G}$ on the parameter space Θ via the correspondence $gX \sim P_{\overline{g}\theta}$. Suppose, moreover, that $\overline{G}$ induces a group G^* on $\mathcal{H}$ via $h(\overline{g}\theta) = g^*h(\theta)$. If $\delta : \mathcal{X} \to \mathcal{H}$ yields an estimator $\delta(X)$ of $h(\theta)$, then we expect to use $g^* \circ \delta(X)$ or $\delta(gX)$ to estimate $h(\overline{g}\theta)$. Thus equivariance is just the requirement that $g^* \circ \delta(X) = \delta(gX)$.

It is fairly straightforward to relate this to the testing situation, in which case $h(\theta) = 1_{\Theta_K}(\theta)$ and the induced group G^* reduces to the trivial group $G = \{e\}$.

Here are two examples of equivariance in estimation:

Example 1. Location Suppose that $X = (X_1, \ldots, X_n)$ where the X_i's are i.i.d. $N(\theta, \sigma^2)$ where $\sigma^2 > 0$ is known. We want to estimate $h(\theta) = \theta$. If

$G = \{g_c : g_c(x) = x + c\underline{1},\ c \in R\}$, then the induced group on the parameter space is $\overline{G} = \{\overline{g}_c : g_c(\theta) = \theta + c,\ c \in R\}$, and this is also the group G^* in the discussion above. Note that for the usual estimator $\delta(X) = \overline{X} = n^{-1}\sum_1^n X_i$ we have

$$\delta(g_c X) = \overline{X} + c = g_c^*(\overline{X}),$$

i.e. $\delta = \overline{X}$ is (location) equivariant.

Example 2. Scale Suppose that $X = (X_1, \ldots, X_n)$ where the X_i's are i.i.d. $N(0, \theta^2)$ where $\theta > 0$ is unknown. We want to estimate $h(\theta) = \theta^2$. If $G = \{g_c :\ g_c(x) = cx,\ x \in R^n,\ c > 0\}$, then the induced group on the parameter space $\Theta = \{\theta :\ \theta > 0\}$ is $\overline{G} = \{\overline{g}_c :\ \overline{g}_c(\theta) = c\theta\}$, and in this case the group $G^* = \{g_c^* :\ g_c^*(h) = c^2 h :\ c > 0\}$ since $h(\overline{g}\theta) = (c\theta)^2 = c^2 h(\theta)$. For the natural (consistent) estimator $\delta(X) = S_X^2 = n^{-1}\sum_1^n (X_i - \overline{X})^2$ of $h(\theta) = \theta^2$, we have

$$\delta(g_c(X)) = c^2 S_X^2 = g_c^*(\delta(X)),$$

i.e. $\delta = S_X^2$ is (scale)-equivariant.