Statistics 583, Problem Set 5

Wellner; 4/29/2009

Reading: Chapter 8, sections 8.1- 8.3; Wasserman, Chapters 2-3, pages 13-41.
Due: Wednesday, May 6, 2009
Reminder 1: Midterm exam, Monday, May 4.
Reminder 2: Makeup lecture, Friday, May 8, 11:30 - 12:20 AM, EEB 025.

1. Exercise 3.8.1, Wasserman, page 39. [Hint: the formula given by Wasserman, page 29, is not correct.] Under what additional hypotheses can we establish $\sqrt{n}(T(\mathbb{F}_n) - T(F)) \to N(0, E_F \psi_F^2(X))$? (Here my ψ_F equals Wasserman's L_F.)

2. (a) Exercise 2.7.9, Wasserman, page 24.
(b) What additional hypotheses are needed to show that $\sqrt{n}(T(\mathbb{F}_n) - T(F))$ is asymptotically normal for this particular functional $T(F)$?
Reminder: This exercise gives the same result as we derived last Fall in Stat 581.

3. Suppose that we observe $X_1, \ldots, X_n$ i.i.d. P on $\mathbb{R}^+ = [0, \infty)$ and assume that $P \in \mathcal{P}_0 \equiv \{P_\theta : \ (dP_\theta/d\lambda) = p_\theta, \ \theta \in \Theta\}$ where $\theta = (\alpha, \beta) \in (0, \infty)^2$ and $p_\theta = p_{\alpha,\beta}$ is the Weibull density given by $p_\theta(x) = (\beta/\alpha)(x/\alpha)^{\beta-1} \exp(-(x/\alpha)^\beta) 1_{(0,\infty)}(x)$. From Lehmann and Romano, TPE, Example 6.6.1 (page 468) and problems 6.6.1 - 6.6.3 (page 509), we know that the maximum likelihood estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ exists and is unique if $0 < X_{(1)} < X_{(n)}$.
(a) If, in fact, $P \notin \mathcal{P}_0$, to what function of P do you expect $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ converges (in probability)? [Hint: use the development in example 6.6.1 of Lehmann and Romano to show that the solution of the population version of the score equations (with sampling from $P \notin \mathcal{P}$) leads to $\alpha(P) = \{E_P(X^\beta)\}^{1/\beta}$ where β is the solution of
$$\frac{E_P(X^\beta \log X)}{E_P X^\beta} - \frac{1}{\beta} = E_P(\log X),$$
assuming that $E_P(X^\beta |\log X|) < \infty$.]
(b) Show heuristically that $\theta(P) = (\alpha(P), \beta(P))$ minimizes $K(P, P_\theta)$ over Θ.
(c) In particular, if P has Gamma$(4,1)$ density $p(x) = (x^3 e^{-x}/3!) 1_{(0,\infty)}(x)$ find $(\alpha, \beta) = (\alpha(P), \beta(P))$ corresponding to the "best-fitting" member of the Weibull family $P_{(\alpha(P),\beta(P))}$. Plot both p and $p_{(\alpha(P),\beta(P))}$ as functions of x.

4. Suppose that $\hat{\theta}_n$ is the MLE for the Weibull family as in problem 1 above, and that $P \notin \mathcal{P}_0$. Heuristically we expect that
$$\sqrt{n}(\hat{\theta}_n - \theta(P)) \to_d N_2(0, \Sigma(P)) \qquad (1)$$

for some covariance matrix $\Sigma = \Sigma(P)$ as $n \to \infty$.
(a) What is the form of Σ that you expect in (1)?
(b) What methods could be used to make these heuristics precise?

5. **Optional bonus problem:** Consider the class of all two-dimensional balls $\mathcal{C}$ in $\mathbb{R}^2$: i.e.

$$\mathcal{C} = \{C: \ C = \{x \in \mathbb{R}^2 : \ \|x - x_0\| \le r\}, \ x_0 \in \mathbb{R}^2, \ r > 0\}.$$

(This is exactly the class $\mathcal{A}$ of two-dimensional spheres described in Wasserman's Exercise 2.7.14, page 25, with my x_0, r being Wasserman's $(a, b), c$.)
(a) Find the VC dimension of $\mathcal{A}$.
(b) Suppose that $X_1, \ldots, X_n$ are i.i.d. P on $\mathbb{R}^2$, and let $\mathbb{P}_n$ be the empirical measure of the X_i's. Use the Vapnik-Chervonenkis inequality given in Wasserman's theorem 2.41 to describe a (conservative) $1 - \alpha = .95$ simultaneous confidence set for all the probabilities $\{P(C) : \ C \in \mathcal{C}\}$.