

## Statistics 583, Problem Set 9 Solutions

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1. (a) Verify the variance part of Theorem 6.9, Wasserman, page 129.
- (b) Verify the proof of (6.12) - (6.14) in Theorem 6.11, Wasserman, page 129.  
Two questions about the proof:
  - (i) Is the hypothesis “ $f'$  is absolutely continuous” (i.e.  $f'(x) = \int_0^x f''(y)dy$  for a function  $f''$ ) used in the proof, or should the hypothesis really be that  $f$  is absolutely continuous; i.e.  $f(x) = \int_0^x f'(x)dx$  for a function  $f'$  (satisfying  $\int_0^1 f'(y)^2 dy < \infty$ )?
  - (ii) Is the  $o(1/n)$  term on page 142, line -8, correct? Does this have any effect on (6.12)?
- (c) Can you find the limiting distribution of  $n^{1/3}(\hat{f}_n(x) - f(x))$  under hypotheses similar to those used in (a corrected version of) Theorem 6.11?

**Solution:** (a) Since

$$\hat{f}_n(x) = \sum_{j=1}^m \frac{\hat{p}_j}{h} 1_{B_j}(x) \quad \text{and} \quad \text{Var}(\hat{p}_j) = \frac{p_j(1-p_j)}{n},$$

it follows that

$$\text{Var}(\hat{f}_n(x)) = \sum_{j=1}^n \frac{p_j(1-p_j)}{nh^2} 1_{B_j}(x).$$

(b) First,

$$\begin{aligned} R(\hat{f}_n, f) &= E \int (\hat{f}_n(x) - f(x))^2 dx \\ &= \int \text{Var}(\hat{f}_n(x)) dx + \int \text{bias}_n^2(\hat{f}_n(x)) dx, \end{aligned}$$

so it suffices to compute these two contributions to the risk separately. First the

integrated variance term: by the calculation in part (a) above,

$$\begin{aligned}
\int \text{Var}(\widehat{f}_n(x))dx &= \int \sum_{j=1}^m \frac{p_j(1-p_j)}{nh^2} 1_{B_j}(x)dx \\
&= \frac{1}{nh^2} \sum_{j=1}^m p_j \int_{B_j} (x)dx - \frac{1}{nh^2} \sum_{j=1}^m p_j^2 \int_{B_j} dx \\
&= \frac{1}{nh} - \frac{1}{nh} \sum_{j=1}^n p_j^2 \\
&= \frac{1}{nh} - \frac{1}{nh} \sum_{j=1}^n h^2 f(x_j)^2 \quad \text{since } p_j = hf(x_j) \text{ for some } x_j \in B_j \\
&= \frac{1}{nh} - \frac{1}{n} \sum_{j=1}^n hf(x_j)^2 \\
&= \frac{1}{nh} - \frac{1}{n} \sum_{j=1}^n \left( \int_{B_j} f^2(x)dx + o(h) \right) \\
&= \frac{1}{nh} - \frac{1}{n} \left( \int_0^1 f^2(x)dx + o(1) \right) = \frac{1}{nh} + O(n^{-1}).
\end{aligned}$$

To compute the bias term, we first use absolute continuity of  $f'$  to expand  $f$  as

$$f(y) = f(x) + (y-x)f'(x) + \frac{1}{2}f''(\tilde{x})(y-x)^2$$

for some  $\tilde{x}$  with  $|\tilde{x} - x| \leq |y - x|$ . Therefore

$$\begin{aligned}
p_j &= \int_{B_j} f(y)dy = \int_{B_j} \left( f(x) + (y-x)f'(x) + \frac{1}{2}f''(\tilde{x})(y-x)^2 \right) dy \\
&= hf(x) + f'(x) \frac{1}{2}(y-x)^2 \Big|_{(j-1)/m}^{j/m} + O(h^3) \\
&= hf(x) + hf'(x) (h(j-1/2) - x) + O(h^3).
\end{aligned}$$

Thus

$$\begin{aligned}
\text{bias}_n(\widehat{f}(x)) &\equiv b(x) = E\widehat{f}_n(x) - f(x) = \frac{p_j}{h} - f(x) \\
&= \frac{1}{h} (hf(x) + hf'(x) (h(j-1/2) - x) + O(h^3)) - f(x) \\
&= f'(x) (h(j-1/2) - x) + O(h^2),
\end{aligned}$$

and this yields

$$\begin{aligned}
\int_{B_j} b^2(x) dx &= \int_{B_j} f'(x)^2 (h(j-1/2) - x)^2 dx + O(h^4) \\
&= f'(\tilde{x}_j)^2 \int_{B_j} (h(j-1/2) - x)^2 dx + O(h^4) \\
&= f'(\tilde{x}_j)^2 \frac{h^3}{12} + O(h^4)
\end{aligned}$$

for some  $\tilde{x}_j \in B_j$  and where we have computed

$$\int_{B_j} (h(j-1/2) - x)^2 dx = -\frac{1}{3} (h(j-1/2) - x) \Big|_{(j-1)/m}^{j/m} = \frac{h^3}{12}.$$

Therefore

$$\begin{aligned}
\int_0^1 b^2(x) dx &= \sum_{j=1}^m \int_{B_j} b^2(x) dx \\
&= \sum_{j=1}^m \left( f'(\tilde{x}_j)^2 \frac{h^3}{12} + O(h^4) \right) \\
&= \frac{h^2}{12} \sum_{j=1}^m h (f'(\tilde{x}_j))^2 + O(h^3) \\
&= \frac{h^2}{12} \int_0^1 f'(x)^2 dx + o(h^2).
\end{aligned}$$

Putting these pieces together yields

$$R(\hat{f}_n, f) = \frac{1}{nh} + O(n^{-1}) + \frac{h^2}{12} \int_0^1 f'(x)^2 dx + o(h^2).$$

Thus it seems that the term  $o(n^{-1})$  in Wasserman's (6.12) should be  $O(n^{-1})$ . For a very careful treatment of this problem, see Freedman and Diaconis, "On the histogram as a density estimator:  $L_2$  theory", *Z. Wahrschein. verw. Geb.* **57** (1981), 453 - 476.

(c) If we take the bandwidth  $h = h_n = Cn^{-1/3}$ , then

$$\begin{aligned}
n^{1/3}(\hat{f}_n(x) - f(x)) &= n^{1/3}(\hat{f}_n(x) - E\hat{f}_n(x) + n^{1/3}(E\hat{f}_n(x) - f(x))) \\
&= n^{1/3}(\hat{f}_n(x) - E\hat{f}_n(x)) + n^{1/3}b_n(x)
\end{aligned}$$

where, from our calculations in (b),

$$\begin{aligned} n^{1/3}b_n(x) &= f'(x)n^{1/3}(h_n(j-1/2) - x) + O(h_n^2) \\ &\begin{cases} \leq f'(x)n^{1/3}(h_n/2) + O(h_n^2) = f'(x)C/2 + O(h_n^2), \\ \geq f'(x)n^{1/3}(-h_n/2) + O(h_n^2) = -f'(x)C/2 + O(h_n^2) \end{cases} \end{aligned}$$

Thus it follows that

$$-f'(x)C/2 \leq \liminf_n n^{1/3}b_n(x) \leq \limsup_n n^{1/3}b_n(x) \leq f'(x)C/2.$$

On the other hand for the random term  $n^{1/3}(\widehat{f}_n(x) - E\widehat{f}_n(x))$  we have

$$\begin{aligned} n^{1/3}(\widehat{f}_n(x) - E\widehat{f}_n(x)) &= \frac{n^{1/3}}{h_n}(\widehat{p}_j - p_j)1_{B_j}(x) \\ &= n^{2/3}C(\widehat{p}_j - p_j)1_{B_j}(x) \\ &= \frac{1}{C} \sum_{i=1}^n n^{-1/3}(1_{B_j}(X_i) - p_j) \\ &\equiv \frac{1}{C} \sum_{i=1}^n X_{ni}. \end{aligned}$$

Now  $\{X_{ni} : i = 1, \dots, n\}$  satisfy  $E(X_{ni}) = 0$ ,

$$\begin{aligned} \sigma_{ni}^2 = \text{Var}(X_{ni}) &= n^{-2/3}p_j(1-p_j) = n^{-2/3}f(x_j^*)h - n^{-2/3}(f(x_j^*)h)^2 \\ &= Cf(x_j^*)n^{-1} - C^2f(x_j^*)^2n^{-4/3}, \end{aligned}$$

so that

$$\sum_{i=1}^n \sigma_{ni}^2 = Cf(x_j^*) - O(h_n) \rightarrow Cf(x)$$

assuming that  $f$  is continuous at  $x$ . Furthermore,

$$\frac{1}{\sigma_n^2} \sum_{i=1}^n E|X_{ni}|^2 1\{|X_{ni}| > \epsilon\sigma_n\} \rightarrow 0$$

for every  $\epsilon > 0$  since

$$|X_{ni}| \leq n^{-1/3} \leq \epsilon Cf(x)/2$$

for all  $n$  sufficiently large if  $f(x) > 0$ . Thus we conclude from the Lindeberg-Feller CLT that

$$\frac{\sum_{i=1}^n n^{-1/3}(1_{B_j}(X_i) - p_j)}{\sigma_n} \rightarrow_d N(0, 1),$$

and hence that

$$\begin{aligned} \sum_{i=1}^n n^{-1/3}(1_{B_j}(X_i) - p_j) &\rightarrow_d N(0, Cf(x)), \\ n^{1/3}(\widehat{f}_n(x) - E\widehat{f}_n(x)) &\rightarrow_d \frac{1}{C}N(0, Cf(x)) = N(0, f(x)/C). \end{aligned}$$

Therefore although we cannot conclude that  $n^{1/3}(\widehat{f}_n(x) - f(x)) \rightarrow_d$ , it does follow that

$$n^{1/3}(\widehat{f}_n(x) - f(x)) = n^{1/3}(\widehat{f}_n(x) - E\widehat{f}_n(x)) + n^{1/3}b_n(x)$$

where the random term converges in distribution to  $N(0, f(x)/C)$  and the bias term is bounded between  $-f'(x)C/2$  and  $f'(x)C/2$ .

If we choose the bandwidth to be just slightly smaller than order  $n^{-1/3}$ , say  $h_n = C(n \log n)^{-1/3}$  then the bias term is small relative to  $\sqrt{nh_n} = \sqrt{nn^{-1/3}(\log n)^{-1/3}} = n^{1/3}(\log n)^{-1/6}$  (since  $\sqrt{nh_n}b_n(x) = O(n^{1/3}(\log n)^{-1/6}n^{-1/3}(\log n)^{-1/3}) = O((\log n)^{-1/2})$ ), and the random term still satisfies

$$\sqrt{nh_n}(\widehat{f}_n(x) - E\widehat{f}_n(x)) \rightarrow_d N(0, Cf(x)),$$

so we can conclude that

$$\sqrt{nh_n}(\widehat{f}_n(x) - f(x)) \rightarrow_d N(0, f(x)/C)$$

with this choice of the bandwidth.

An interesting comparison is to investigate the behavior of a kernel estimator with a “boxcar kernel”,  $K(x) = 2^{-1}1_{[-1,1]}(x)$  in this same situation with just one continuous derivative  $f'$ : then with  $h_n = Cn^{-1/3}$  I find that

$$n^{1/3}(\widehat{f}_n(x) - f(x)) \rightarrow_d N(Cf'(x)/2, f(x) \int K^2(z)dz/C).$$

Note that as  $C$  increases the asymptotic bias grows, but the asymptotic variance decreases.

2. (a) Consider the kernel density estimator defined in (6.26), Wasserman, page 132. Show that if the density  $f$  and the kernel  $k$  satisfy the hypotheses of Wasserman’s theorem 6.28, page 133, and  $h = h_n$  satisfies the hypotheses of Theorem 6.27, then for fixed  $x \in \mathbb{R}$ ,

$$\sqrt{nh_n}(\widehat{f}_n(x) - E\widehat{f}_n(x)) \rightarrow_d N\left(0, f(x) \int k^2(x)dx\right).$$

(b) Under what restriction on  $h_n$  does it follow (from (a) together with further analysis of the bias) that

$$\sqrt{nh_n}(\hat{f}_n(x) - f(x)) \rightarrow_d N\left(0, f(x) \int k^2(x)dx\right)?$$

(c) If  $h_n = cn^{-1/5}$  and the hypotheses of (a) hold, find the limiting distribution of  $\sqrt{nh_n}(\hat{f}_n(x) - f(x))$ .

(d) Under the same assumptions as in (c), find the limiting distribution of  $\sqrt{nh_n}(\sqrt{\hat{f}_n(x)} - \sqrt{f(x)})$ .

(e) Suppose that  $x, y \in \mathbb{R}$  with  $x < y$ . Find the joint limiting distribution of  $(\sqrt{nh_n}(\hat{f}_n(x) - f(x)), \sqrt{nh_n}(\hat{f}_n(y) - f(y)))$  under the assumptions in (b) and (c).

**Solution:** (a) First write

$$\begin{aligned} & \sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) \\ &= \sqrt{nh_n} \left( \frac{1}{nh_n} \sum_{i=1}^n k((x - X_i)/h_n) - \frac{1}{h_n} Ek((x - X_1)/h_n) \right) \\ &= \sum_{i=1}^n \frac{1}{\sqrt{nh_n}} (k((x - X_i)/h_n) - Ek((x - X_1)/h_n)) \\ &\equiv \sum_{i=1}^n X_{n,i} \end{aligned}$$

where the  $X_{n,i}$ 's are independent and identically distributed for each  $n$ , but with a distribution depending on  $n$ . Thus we will use the Lindeberg-Feller CLT. By easy calculations,  $E(X_{n,i}) = 0$ , and

$$\sigma_{n,i}^2 = \text{Var}(X_{n,i}) = \frac{1}{nh_n} \int \left( k((x - y)/h_n) - \int k((x - v)/h_n)f(v)dv \right)^2 f(y)dy,$$

so

$$\begin{aligned} \sigma_n^2 &= \sum_{i=1}^n \sigma_{n,i}^2 \\ &= \frac{1}{h_n} \int \left( k((x - y)/h_n) - \int k((x - v)/h_n)f(v)dv \right)^2 f(y)dy \\ &= \int k(z)^2 f(x - zh_n)dz - h_n \left( \int k(z)f(x - zh_n)dz \right)^2 \\ &\rightarrow f(x) \int k(z)^2 dz \end{aligned}$$

by the dominated convergence theorem (if, for example,  $f$  is bounded). It remains only to verify the Lindeberg condition:

$$\frac{1}{\sigma_n^2} \sum_{i=1}^n E\{|X_{n,i}|^2 \mathbf{1}\{|X_{n,i}| > \epsilon \sigma_n\}\} \rightarrow 0$$

as  $n \rightarrow \infty$  for every  $\epsilon > 0$ . But

$$\begin{aligned} X_{ni} &\equiv \frac{1}{\sqrt{nh_n}} \left\{ k\left(\frac{x - X_i}{h_n}\right) - \int k\left(\frac{x - v}{h_n}\right) f(v) dv \right\} \\ &= \frac{1}{\sqrt{nh_n}} k\left(\frac{x - X_i}{h_n}\right) - \frac{h_n}{\sqrt{nh_n}} \int k(z) f(x - zh_n) dz \\ &\equiv Y_{ni} - \sqrt{\frac{h_n}{n}} (f(x) + o(1)). \end{aligned}$$

Thus

$$\begin{aligned} |X_{ni}| &\leq |Y_{ni}| + \sqrt{\frac{h_n}{n}} (f(x) + o(1)), \\ |X_{ni}|^2 &\leq 2\{Y_{ni}^2 + \frac{h_n}{n} (f(x) + o(1))^2\}, \quad \text{and} \\ \{|X_{ni}| > \epsilon \sigma_n\} &\subset \{|Y_{ni}| \geq (1/2)\epsilon \sigma_n\} \cup \{\sqrt{h_n/n} (f(x) + o(1)) > (1/2)\epsilon \sigma_n\} \\ &= \{|Y_{ni}| \geq (1/2)\epsilon \sigma_n\} \end{aligned}$$

for  $n$  sufficiently large since the second event on the right is empty for large  $n$ . It follows that

$$\begin{aligned} &E|X_{n,i}|^2 \mathbf{1}\{|X_{n,i}| > \epsilon \sigma_n\} \\ &\leq 2E|Y_{n,i}|^2 \mathbf{1}\{|Y_{n,i}| > \epsilon \sigma_n/2\} + 2\frac{h_n}{n} (f(x) + o(1))^2 E\mathbf{1}\{|Y_{ni}| \geq (1/2)\epsilon \sigma_n\} \\ &\leq 2E|Y_{n,i}|^2 \mathbf{1}\{|Y_{n,i}| > \epsilon \sigma_n/2\} + \frac{2\frac{h_n}{n} (f(x) + o(1))^2}{((1/2)\epsilon \sigma_n)^2} E|Y_{n,i}|^2 \mathbf{1}\{|Y_{ni}| \geq (1/2)\epsilon \sigma_n\} \\ &= (2 + o(1))E|Y_{n,i}|^2 \mathbf{1}\{|Y_{n,i}| > \epsilon \sigma_n/2\}, \end{aligned}$$

and hence it suffices to show that

$$\sum_{i=1}^n E|Y_{n,i}|^2 \mathbf{1}\{|Y_{n,i}| > \epsilon\} \rightarrow 0 \quad \text{for all } \epsilon > 0.$$

But

$$\begin{aligned}
& \sum_{i=1}^n E|Y_{n,i}|^2 1\{|Y_{n,i}| > \epsilon\} \\
&= \sum_{i=1}^n E \frac{k((x - X_i)/h_n)^2}{nh_n} 1\{k((x - X_i)/h_n) > \epsilon\sqrt{nh_n}\} \\
&= \frac{1}{h_n} \int k((x - y)/h_n)^2 1\{k((x - y)/h_n) > \epsilon\sqrt{nh_n}\} f(y) dy \\
&= \int k^2(z) f(x - zh_n) 1\{k(z) > \epsilon\sqrt{nh_n}\} dz \\
&\rightarrow 0
\end{aligned}$$

by the dominated convergence theorem if  $\|f\|_\infty < \infty$ . Thus by the Lindeberg-Feller CLT it follows that  $\sum_1^n X_{n,i}/\sigma_n \rightarrow N(0, 1)$ . Since  $\sigma_n^2 \rightarrow \int k^2(z) dz$ , the conclusion follows.

(b) Note that the bias is

$$\begin{aligned}
E\hat{f}_n(x) - f(x) &= \int k(z) f(x - zh_n) dz - f(x) = \int k(z) \{f(x - zh_n) - f(x)\} dz \\
&= \int k(z) \frac{1}{2} f''(z_x^*) z^2 h_n^2 dz
\end{aligned}$$

where  $|z_x^* - x| \leq |x - zh_n - x| = |z|h_n$ . Thus if  $f''(x)$  is bounded and  $\sqrt{nh_n} h_n^2 \rightarrow 0$ , the bias term is negligible and

$$\begin{aligned}
\sqrt{nh_n}(\hat{f}_n(x) - f(x)) &= \sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) + \sqrt{nh_n}(E\hat{f}_n(x) - f(x)) \\
&\rightarrow_d N(0, f(x) \int k^2(z) dz) + 0 = N(0, f(x) \int k^2(z) dz).
\end{aligned}$$

Note that  $\sqrt{nh_n} h_n^2 = (nh_n^5)^{1/2} \rightarrow 0$  if and only if  $nh_n^5 \rightarrow 0$ ; i.e. if  $h_n = o(n^{-1/5})$ . Thus  $h_n = cn^{-1/4}$  yields negligible bias when  $f''$  exists (and is sufficiently bounded).

(c) When  $h_n = cn^{-1/5}$ , then the first display in (b) yields

$$\begin{aligned}
\sqrt{nh_n}(E\hat{f}_n(x) - f(x)) &= \sqrt{cn^{2/5}} c^2 n^{-2/5} \frac{1}{2} \int z^2 k(z) f''(z_x^*) dz \\
&\rightarrow \frac{c^{5/2}}{2} f''(x) \int z^2 k(z) dz \equiv B(c, f, k, x)
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\sqrt{nh_n}(\hat{f}_n(x) - f(x)) &= \sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) + \sqrt{nh_n}(E\hat{f}_n(x) - f(x)) \\
&\rightarrow_d N(0, f(x) \int k^2(z)dz) + B(c, f, k) \\
&= N(B(c, f, k, x), f(x) \int k^2(z)dz).
\end{aligned}$$

(d) Under the assumptions in (c), it follows from the delta-method with  $g(v) = v^{1/2}$  that for any  $x$  such that  $f(x) > 0$  we have

$$\begin{aligned}
\sqrt{nh_n}(\sqrt{\hat{f}_n(x)} - \sqrt{f(x)}) &\rightarrow_d \frac{1}{2}f(x)^{-1/2}N\left(B(c, f, k), f(x) \int k^2(z)dz\right) \\
&= N\left(\frac{B(c, f, k, x)}{2\sqrt{f(x)}}, 4^{-1} \int k^2(z)dz\right).
\end{aligned}$$

Note that the variance now depends only on the kernel and not on  $f$ .

(e) To find the joint limiting distribution of  $(\sqrt{nh_n}(\hat{f}_n(x) - f(x)), \sqrt{nh_n}(\hat{f}_n(y) - f(y)))$  we will use the Cramér - Wold device: let  $(a, b) \in \mathbb{R}^2$ . We will first show that

$$\begin{aligned}
a\sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) + b\sqrt{nh_n}(\hat{f}_n(y) - E\hat{f}_n(y)) \\
\rightarrow_d N\left(0, a^2f(x) \int k^2(z)dz + b^2f(y) \int k^2(z)dz\right)
\end{aligned} \tag{1}$$

for each fixed  $(a, b) \in \mathbb{R}^2$ . This implies that

$$\begin{pmatrix} \sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) \\ \sqrt{nh_n}(\hat{f}_n(y) - E\hat{f}_n(y)) \end{pmatrix} \rightarrow_d N_2\left(0, \begin{pmatrix} f(x) & 0 \\ 0 & f(y) \end{pmatrix} \int k^2(z)dz\right) \tag{2}$$

and will yield the desired results after analyzing the relevant bias terms. But

$$\begin{aligned}
&a\sqrt{nh_n}(\hat{f}_n(x) - E\hat{f}_n(x)) + b\sqrt{nh_n}(\hat{f}_n(y) - E\hat{f}_n(y)) \\
&= \frac{1}{nh_n} \sum_{i=1}^n \left\{ a \left\{ k\left(\frac{x - X_i}{h_n}\right) - \int k\left(\frac{x - v}{h_n}\right) f(v)dv \right\} \right. \\
&\quad \left. + b \left\{ k\left(\frac{y - X_i}{h_n}\right) - \int k\left(\frac{y - v}{h_n}\right) f(v)dv \right\} \right\} \\
&\equiv \sum_{i=1}^n \tilde{X}_{n,i}
\end{aligned}$$

where

$$\begin{aligned}\tilde{X}_{n,i} &\equiv \frac{1}{nh_n} \left\{ a \left\{ k\left(\frac{x-X_i}{h_n}\right) - \int k\left(\frac{x-v}{h_n}\right) f(v)dv \right\} \right. \\ &\quad \left. + b \left\{ k\left(\frac{y-X_i}{h_n}\right) - \int k\left(\frac{y-v}{h_n}\right) f(v)dv \right\} \right\}\end{aligned}$$

We compute  $E\tilde{X}_{n,i} = 0$  as in (a), but now

$$\begin{aligned}\sigma_{ni}^2 &= Var(\tilde{X}_{n,i}) \\ &= (nh_n)^{-1} \left\{ a^2 n Var\{k((x-X_1)/h_n) + b^2 Var((y-X_1)/h_n) \right. \\ &\quad \left. + 2ab Cov(k((x-X_1)/h_n), k((y-X_1)/h_n)) \right\}\end{aligned}$$

so that

$$\begin{aligned}\sigma_n^2 &= \sum_{i=1}^n \sigma_{ni}^2 \\ &= h_n^{-1} \left\{ a^2 \int \left( k((x-w)/h_n) - \int k((x-v)/h_n) f(v)dv \right)^2 f(w)dw \right. \\ &\quad \left. + b^2 \int \left( k((y-w)/h_n) - \int k((y-v)/h_n) f(v)dv \right)^2 f(w)dw \right. \\ &\quad \left. + 2ab \int \left( k((x-w)/h_n) - \int k((x-v)/h_n) f(v)dv \right) \right. \\ &\quad \left. \left( k((y-w)/h_n) - \int k((y-v)/h_n) f(v)dv \right) f(w)dw \right\} \\ &\rightarrow a^2 f(x) \int k^2(z)dz + b^2 f(y) \int k^2(z)dz + 0\end{aligned}$$

where the argument for the first two terms is exactly as in (a), and the third term converges to 0 since

$$\begin{aligned}&h_n^{-1} \int \left( k((x-w)/h_n) - \int k((x-v)/h_n) f(v)dv \right) \\ &\quad \left( k((y-w)/h_n) - \int k((y-v)/h_n) f(v)dv \right) f(w)dw \\ &= h_n^{-1} \int \left( k((x-w)/h_n) - h_n^2(f(x) + o(1)) \right) \\ &\quad \left( k((y-w)/h_n) - h_n^2(f(y) + o(1)) \right) f(w)dw \\ &= \int k\left(z - \frac{y-x}{h_n}\right) k(z) f(y-zh_n) dz + O(h_n) \\ &\rightarrow \int k(-\infty) k(z) dz f(y) = 0\end{aligned}$$

by the dominated convergence theorem if  $f$  and  $k$  are bounded using  $(y-x)/h_n \rightarrow \infty$  since  $y-x > 0$  and  $h_n \rightarrow 0$ . Since  $|\tilde{X}_{n,i}| \leq |a||X_{n,i}(x)| + |b||X_{n,i}(y)|$ , verification of the Lindeberg condition proceeds as in (a). Thus by the Lindeberg-Feller CLT, (1) holds, and this in turn yields (2).

3. (a) Wasserman, problem 6.9.3, page 143.  
 (b) Show that (6.35) on page 136 holds.

**Solution:** (a) First the case of kernel density estimates: here

$$J(h) = \int \hat{f}_n(x)^2 dx - 2 \int \hat{f}_n(x)f(x)dx, \quad \text{and}$$

$$\hat{J}(h) = \int \hat{f}_n(x)^2 dx - \frac{2}{n} \sum_{i=1}^n \hat{f}_{(-i)}(X_i),$$

so to show that  $E\{\hat{J}(h)\} = E\{J(h)\}$ , it suffices to show that

$$E \left\{ \frac{1}{n} \sum_{i=1}^n \hat{f}_{(-i)}(X_i) \right\} = E \left\{ \int \hat{f}_n(x)f(x)dx \right\}. \quad (3)$$

Now the right side in the last display is

$$\begin{aligned} E \left\{ \int \hat{f}_n(x)f(x)dx \right\} &= \int E\{\hat{f}_n(x)\}f(x)dx \\ &= \iint \frac{1}{h}k\left(\frac{x-y}{h}\right) f(y)dyf(x)dx, \end{aligned}$$

while the left side is, by conditioning on  $X_i$ ,

$$\begin{aligned} E \left\{ \frac{1}{n} \sum_{i=1}^n \hat{f}_{(-i)}(X_i) \right\} &= \frac{1}{n} \sum_{i=1}^n E \left\{ E(\hat{f}_{(-i)}(X_i)|X_i) \right\} \\ &= \frac{1}{n} \sum_{i=1}^n E \left\{ \int \frac{1}{h}k\left(\frac{X_i-x}{h}\right) f(x)dx \right\} \\ &= \iint \frac{1}{h}k\left(\frac{y-x}{h}\right) f(y)dyf(x)dx \\ &= \iint \frac{1}{h}k\left(\frac{x-y}{h}\right) f(y)dyf(x)dx \end{aligned}$$

by Fubini's theorem. Comparing the last two displays yields the claim.

In the case of histogram estimators, it again suffices to show that (3) holds. But

$$\begin{aligned} E \left\{ \int \widehat{f}_n(x) f(x) dx \right\} &= \int E \{ \widehat{f}_n(x) \} f(x) dx = \int \sum_{j=1}^m \frac{p_j}{h} 1_{B_j}(x) f(x) dx \\ &= \sum_{j=1}^m \frac{p_j^2}{h}, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} E \left\{ \frac{1}{n} \sum_{i=1}^n \widehat{f}_{(-i)}(X_i) \right\} &= E \left\{ E \left\{ n^{-1} \sum_{i=1}^n \widehat{f}_{(-i)}(X_i) \mid X_i \right\} \right\} \\ &= E \left\{ n^{-1} \sum_{i=1}^n \sum_{j=1}^m \frac{p_j}{h} 1_{B_j}(X_i) \right\} \\ &= \sum_{j=1}^m \frac{p_j}{h} \int_{B_j} f(x) dx = \sum_{j=1}^m \frac{p_j^2}{h}, \end{aligned}$$

and hence we conclude that (again)  $E\{\widehat{J}(h)\} = E\{J(h)\}$ .

(b) The claimed identity is

$$\widehat{J}(h) = \frac{1}{n^2 h} \sum_i \sum_j k^* \left( \frac{X_i - X_j}{h} \right) + \frac{2}{nh} k(0) + O(n^{-2})$$

where  $k^*(x) = k^{(2)}(x) - 2k(x)$  and  $k^{(2)}(z) = \int k(z-y)k(y)dy$ . To check this, write

$$\widehat{f}_n^2(x) = \frac{1}{n^2 h^2} \sum_i \sum_j k \left( \frac{x - X_i}{h} \right) k \left( \frac{x - X_j}{h} \right)$$

so that, with  $k^{(2)}(z) = \int k(z-y)k(y)dy$ ,

$$\begin{aligned} \int \widehat{f}_n^2(x) dx &= \frac{1}{n^2 h^2} \sum_i \sum_j \int k \left( \frac{x - X_i}{h} \right) k \left( \frac{x - X_j}{h} \right) dx \\ &= \frac{1}{n^2 h^2} \sum_i \sum_j \int k \left( \frac{X_j + hy - X_i}{h} \right) k(y) h dy \\ &\quad \text{by the change of variables } (x - X_j)/h = y, \quad x = X_j + hy \\ &= \frac{1}{n^2 h} \sum_i \sum_j \int k \left( y - \frac{X_i - X_j}{h} \right) k(y) dy \\ &= \frac{1}{n^2 h} \sum_i \sum_j k^{(2)} \left( \frac{X_i - X_j}{h} \right) \end{aligned}$$

if we assume that  $k(z) = k(-z)$ .

Similarly,

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^n \widehat{f}_{(-i)}(X_i) &= \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} \frac{1}{(n-1)h} k\left(\frac{X_i - X_j}{h}\right) \\
&= \frac{1}{n(n-1)h} \sum_i \sum_j k\left(\frac{X_i - X_j}{h}\right) - \frac{1}{n-1} k(0) \\
&= \frac{1}{n^2 h} \sum_i \sum_j k\left(\frac{X_i - X_j}{h}\right) - \frac{1}{n} k(0) + O_p(n^{-1})
\end{aligned}$$

Putting these pieces together yields the claimed identity, but with a remainder term which is  $O_p(n^{-1})$  rather than  $O_p(n^{-2})$ . Thus it seems that the computational formula should really read as follows:

$$\widehat{J}(h) = \frac{1}{n^2 h} \sum_i \sum_j k^{(2)}\left(\frac{X_i - X_j}{h}\right) - \frac{2}{n(n-1)h} \sum_{i=1}^n \sum_{j \neq i} k\left(\frac{X_i - X_j}{h}\right).$$