Statistics 583, Problem Set 7 Solutions

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1. The expression for the jackknife variance estimator for the median, in the display (1) on page 11 (3rd line from the bottom) in chapter 8 was derived under the assumption $n = 2m$ and that $T(\mathbb{F}_n) = X_{(m)}$ if $n = 2m-1$, $T(\mathbb{F}_n) = (X_{(m)} + X_{(m+1)})/2$ if $n = 2m$.
(a) Derive the first equality in (1), page 11, using this definition of the sample median.
(b) Derive versions of the development in (1), page 11, using $T(F) = F^{-1}(1/2)$ (strictly). Does the asymptotic result in (1) still hold? Here is some further explanation of what I mean by "strictly" here: let $T_1(\mathbb{F}_n) = X_m$ if $n = 2m-1$, $T_1(\mathbb{F}_n) = (X_{(m)} + X_{(m+1)})/2$ if $n = 2m$. This is one common definition of the median, and this is the definition used in (a). Let $T_2(\mathbb{F}_n) = \mathbb{F}_n^{-1}(1/2)$. This is my favorite definition of the median. Note that $T_2(\mathbb{F}_n) = T_1(\mathbb{F}_n)$ if $n = 2m-1$, but $T_2(\mathbb{F}_n) \neq T_1(\mathbb{F}_n)$ if $n = 2m$. (What is the value of $T_2(\mathbb{F}_n)$ in this case?) T_2 is the definition of the median to be considered in 2(b)!

Solution: Solution: (a). For $n = 2m$,

$$T_{n,i} = \begin{cases} X_{(m+1)} & \text{if } i \le m \\ X_{(m)} & \text{if } i > m \end{cases}$$

and $T_{n,\cdot} = (X_{(m)} + X_{(m+1)})/2$. Hence

$$\begin{aligned} n\widehat{\text{Var}}_n &= (n-1)\Big\{ m(X_{(m+1)} - \frac{1}{2}(X_{(m)} + X_{(m+1)}))^2 \\ &\quad + m(X_{(m)} - \frac{1}{2}(X_{(m)} + X_{(m+1)}))^2 \Big\} \\ &= n(n-1)\{\frac{X_{(m+1)} - X_{(m)}}{2}\}^2 . \qquad (1) \end{aligned}$$

(b). When $n = 2m$ and $T(F) = F^{-1}(1/2)$, we have $T(\mathbb{F}_n) = X_{(m)}$ and $T_{n,i}$ are exactly as in A above. Hence (1) continues to hold.
When $n = 2m-1$, then $T(\mathbb{F}_n) = X_{(m)}$,

$$T_{n,i} = \begin{cases} X_{(m)} & \text{if } i \le m-1 \\ X_{(m-1)} & \text{if } i \ge m \end{cases} ,$$

and $T_{n,\cdot} = \{(m-1)X_{(m)} + mX_{(m-1)}\}/(2m-1)$. Therefore

$$\begin{aligned} n\widehat{\mathrm{Var}}_n &= (n-1)\left\{(m-1)\{X_{(m)} - \frac{1}{2m-1}[(m-1)X_{(m)} + mX_{(m-1)}]\}^2 \right. \\ &\quad \left. + \ m\{X_{(m-1)} - \frac{1}{2m-1}[(m-1)X_{(m)} + mX_{(m-1)}]\}^2\right\} \\ &= \frac{(n-1)^2(n+1)}{n}\left\{\frac{X_{(m)} - X_{(m-1)}}{2}\right\}^2 \\ &\to_d \frac{1}{4f^2(F^{-1}(1/2))}\left(\frac{\chi_2^2}{2}\right)^2 \end{aligned}$$

just as before.

Remark: The only case left out in (a) and (b) is that of an odd sample size, $n = 2m-1$ in part (a). In this case,

$$T_{n,i} = \begin{cases} (X_{(m)} + X_{(m+1)})/2 & \text{if } i \le m-1 \\ (X_{(m-1)} + X_{(m+1)})/2 & \text{if } i = m \\ (X_{(m-1)} + X_{(m)})/2 & \text{if } i \ge m+1 \end{cases} .$$

Thus

$$\begin{aligned} T_{n,\cdot} &= \frac{1}{n}\left\{\frac{(m-1)}{2}(X_{(m)} + X_{(m+1)}) \right. \\ &\quad \left. + \ \frac{1}{2}(X_{(m-1)} + X_{(m+1)}) + \frac{(m-1)}{2}(X_{(m-1)} + X_{(m)})\right\} . \end{aligned}$$

The analysis from this point proceeds not just by algebra, but by careful grouping of terms and observing which terms are negligible. I will not present a full analysis here, but will record the result:

$$\begin{aligned} n\widehat{\mathrm{Var}}_n &= \frac{(m-1)m^2}{2n^3}\left\{n(X_{(m+1)} - X_{(m-1)})\right\}^2 + o_p(1) \\ &\to_d \frac{1}{4f^2(F^{-1}(1/2))}(\frac{\chi_4^2}{4})^2 \end{aligned}$$

since, with $g \equiv F^{-1}$,

$$n(X_{(m+1)} - X_{(m-1)}) \to_d g'(1/2)W$$

where $W =_d Y_1 + Y_2 \sim \text{Gamma}(2,1)$ for independent exponential rv's $Y_1,\ Y_2$, so that $2W \sim \chi_4^2$. Thus for this definition of the sample median, it is true that $n\widehat{\mathrm{Var}}_n = O_p(1)$ for the full sequence of nonnegative integers n but it converges in distribution to one limit as $n = 2m \to \infty$ and a different limit as $n = 2m-1 \to \infty$.

2. (a) Wasserman, problem 3.8.3, page 39, modified. Show that the claimed expression for v_{boot} given in the display for this problem is incorrect and find the correct expression. Here $v_{boot} = Var_{\mathbb{F}_n}(T_n)$ where $T_n = \overline{X}_n^2$. [Hint: see Dodd and Korn, *The American Statistician* **61** (2007), 127 - 131, and especially their appendix B, pages 130-131. Apparently the formula given by Wasserman in his problem is from Shao and Tu (1995), page 10; as noted by Dodd and Korn, the expression in Shao and Tu is incorrect.]
(b) Explain how the resulting formulas relate to how you would estimate the variance of $\overline{X}_n^2$ via the delta method.

Solution: (a) This is explained quite well in the appendix of the paper by Dodd and Korn (2007).
(b) The first term of the exact finite sample variance expression

$$Var(\overline{X}^2) = \frac{4\mu^2\sigma^2}{n} + \frac{2\sigma^4}{n^2} + \frac{4\mu\mu_3}{n^2} + \frac{\mu_4 - 3\sigma^4}{n^3}$$

corresponds exactly to what we would get from the delta method: with $g(x) = x^2$ we have $g'(x) = 2x$ and hence

$$\sqrt{n}(\overline{X}_n^2 - \mu^2) \to_d g'(\mu)\sigma Z \sim N(0, 4\mu^2\sigma^2)$$

where $Z \sim N(0,1)$. Thus the delta-method estimator of $Var(\overline{X}_n^2)$ is just $4\overline{X}_n^2 S_n^2$ where S_n is the sample variance. The bootstrap estimator of variance refines this (as shown by Dodd and Korn) by correctly capturing the n^{-2} term when $\mu \neq 0$. When $\mu = 0$, then neither the (first order) delta method nor the (nonparametric) bootstrap tells the complete story.

3. Wasserman, problem 3.8.11, page 41: suppose that $X_1, \ldots, X_n$ are i.i.d. Uniform$(0, \theta)$. The MLE $\widehat{\theta}_n$ of θ is $\widehat{\theta} = \max\{X_1, \ldots, X_n\} = X_{(n)}$.
(a) Find the distribution of $\widehat{\theta}_n$. Compare the true distribution of $\widehat{\theta}_n$ to the histograms from the parametric and nonparametric bootstraps.
(b) This is a case where the nonparametric bootstrap does very poorly. Show that for the parametric bootstrap $P^*(\widehat{\theta}_n^* = \widehat{\theta}_n) = 0$, but for the nonparametric bootstrap

$$P^*(\widehat{\theta}_n^* = \widehat{\theta}_n) = 1 - (1 - 1/n)^n \to 1 - e^{-1} \approx .632....$$

Solution: (a) Let F_θ denote the Uniform$(0, \theta)$ distribution. The distribution of $\widehat{\theta}_n$ is just

$$\begin{aligned} P_\theta(\widehat{\theta}_n \le x) &= P_\theta(X_{(n)} \le x) = P_\theta(X_1 \le x, \ldots, X_n \le x) \\ &= P_\theta(X_1 \le x)^n = (1 - x/\theta)^n, \qquad 0 \le x \le \theta, \text{ so that} \\ f_{\widehat{\theta}_n}(x) &= \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} 1_{[0,\theta]}(x). \end{aligned}$$

Thus $\widehat{\theta}_n/\theta \stackrel{d}{=} \xi_{(n)}$, the largest order statistic of a sample $\xi_1, \ldots, \xi_n$ of i.i.d. Uniform$(0,1)$ random variables, and $n(1-\widehat{\theta}_n/\theta) \stackrel{d}{=} n\xi_{(1)} \rightarrow_d Y$ where $Y \sim$ exponential(1).
(b) The parametric bootstrap estimator of

$$K_n(x, F_\theta) \equiv P_\theta(\widehat{\theta}_n \le x) = (1 - x/\theta)^n, \quad 0 \le x \le \theta$$

is

$$K_n(x, F_{\widehat{\theta}}) \equiv P_{\widehat{\theta}}(\widehat{\theta}_n^* \le x) = (1 - x/\widehat{\theta})^n, \quad 0 \le x \le \widehat{\theta}.$$

The nonparametric bootstrap estimator is given by

$$\begin{aligned} K_n(x, \mathbb{F}_n) &= P_{\mathbb{F}_n}(\widehat{\theta}_n^* \le x) = P^*(X_{(n)}^* \le x) \\ &= P^*(X_1^* \le x, \ldots, X_n^* \le x) = P^*(X_1^* \le x)^n = \mathbb{F}_n(x)^n. \end{aligned}$$

Thus

$$\begin{aligned} P_{\mathbb{F}_n}(\widehat{\theta}_n^* = \widehat{\theta}_n) &= P_{\mathbb{F}_n}(X_{(n)}^* = X_{(n)}) = \mathbb{F}_n(X_{(n)})^n - \mathbb{F}_n(X_{(n-1)})^n \\ &= 1 - \left(\frac{n-1}{n}\right)^n = 1 - (1 - 1/n)^n \rightarrow 1 - e^{-1}. \end{aligned}$$

4. On page 12, line 4 of Chapter 8 of the lecture notes, it is claimed that if $E_F|X|^r < \infty$ for some $r > 0$ (and F has positive and continuous density f in a neighborhood of $F^{-1}(1/2)$) then for the median function $T(F) = F^{-1}(1/2)$ we have

$$nVar_F(T(\mathbb{F}_n)) \rightarrow \frac{1/4}{f^2(F^{-1}(1/2))}.$$

Prove (or disprove) this claim.

Solution: More general versions of this problem are discussed in Section 11.4.7, pages 474 - 480, Shorack and Wellner (1986). Unfortunately this material in SW (1986) is based to a substantial extent on an unpublished Stanford Ph.D. dissertation of K. M. Anderson (1982), which is not generally available, and the proofs in SW (1986) are incomplete. In particular, Theorem 4 and Exercise 2 on pages 475 - 476 are not proved there: see the last line of Section 11.4.7, page 480. The solution below relies on inequalities derived in Wellner (1977), *Ann. Statist.* **5**, 481-494.

Suppose that $n = 2m$, so that $T(\mathbb{F}_n) = X_{(m)}$. Since

$$\begin{aligned} Var(T(\mathbb{F}_n)) &= E(X_{(m)} - EX_{(m)})^2 \\ &= E(X_{(m)} - F^{-1}(1/2))^2 - (F^{-1}(1/2) - EX_{(m)})^2, \end{aligned}$$

it suffices to show that

$$nE(X_{(m)} - F^{-1}(1/2))^2 \to \frac{1/4}{f^2(F^{-1}(1/2))}, \qquad \text{and}$$
$$n^{1/2}\left\{EX_{(m)} - F^{-1}(1/2)\right\} \to 0.$$

Since we know that $\sqrt{n}(X_{(m)} - F^{-1}(1/2)) \to_d N(0, 1/(4f^2(F^{-1}(1/2)))$, both of these will follow if we show that the sequence of random variables $\{V_n\} \equiv \{n^{1/2}|X_{(m)} - F^{-1}(1/2)|\}_{n\ge 1}$ is uniformly square integrable; i.e. if we show that

$$\limsup_{n\to\infty} E|V_n|^2 1_{[|V_n|\ge\lambda]} \to 0 \qquad \text{as} \ \ \lambda \to \infty.$$

This would follow easily from

$$\limsup_{n\to\infty} E|V_n|^s < \infty \tag{2}$$

for any $s > 2$. Now it follows from Wellner (1977) (see also Shorack and Wellner (1986), page 456) that, with $p_m = m/(2n+1)$,

$$E|\xi_{(m)} - p_m|^s \le C_s(p_m q_m/n)^{s/2} \le C_s(m/n^2)^{s/2} = C_s\left(\frac{1}{2n}\right)^{s/2},$$

where $C_s = 1 + 2\cdot 10^s\Gamma(s+1)$. Equivalently,

$$E\{|n^{1/2}(\xi_{(m)} - p_m)|^s\} \le C_s(1/2)^{s/2}. \tag{3}$$

Now write $Q = F^{-1}$ and, with $\xi_1, \ldots, \xi_n$ i.i.d. Uniform$(0,1)$,

$$\begin{aligned} V_n &= \sqrt{n}(X_{(m)} - F^{-1}(1/2)) \stackrel{d}{=} \sqrt{n}(Q(\xi_{(m)}) - Q(1/2)) \\ &= Q'(\xi^*)\sqrt{n}(\xi_{(m)} - 1/2) \\ &\equiv Q'(\xi^*)U_n \end{aligned}$$

where $|\xi^* - 1/2| \le |\xi_{(m)} - 1/2|$ since the derivative Q' exists and is continuous in a neighborhood of $1/2$. Let $s > 2$, $0 < M < \infty$, and $\epsilon > 0$. Then we have

$$\begin{aligned} E|V_n|^s &= E|V_n|^s 1\{|V_n| \le M\} + E|V_n|^s 1\{|V_n| > M\} \\ &\le M^s + E|V_n|^s 1\{|V_n| > M\} \equiv I + II_n. \end{aligned}$$

Now we bound II_n in several steps:

$$\begin{aligned} II_n &= E|V_n|^s 1\{|V_n| > M\} \\ &\le E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| \le M\}\} + E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| > M\}\} \\ &\le \left(M \sup_{u:|u-1/2|\le M/\sqrt{n}} |Q'(u)|\right)^s + E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| > M\}\} \\ &\equiv III_n + IV_n \end{aligned}$$

since on $\{|U_n| \le M\}$,

$$|V_n| \le |Q'(\xi^*)||U_n| \le M|Q'(\xi^*)| \le M \sup_{u:|u-1/2|\le M/\sqrt{n}} |Q'(u)|.$$

Now we bound IV_n in two steps: for n so large that $\epsilon\sqrt{n} > M$,

$$\begin{aligned}
IV_n &= E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| > M\}\} \\
&= E\{|V_n|^s 1\{|V_n| > M\} 1\{M < |U_n| \le \epsilon\sqrt{n}\}\} + E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| > \epsilon\sqrt{n}\}\} \\
&\le \left(\sup_{u:|u-1/2|\le\epsilon} |Q'(u)|\right)^s E|U_n|^s 1\{|U_n| > M\} + E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| > \epsilon\sqrt{n}\}\} \\
&\le \left(\sup_{u:|u-1/2|\le\epsilon} |Q'(u)|\right)^s E|U_n|^s + E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| > \epsilon\sqrt{n}\}\} \\
&\le \left(\sup_{u:|u-1/2|\le\epsilon} |Q'(u)|\right)^s \cdot C_s (1/2)^{s/2} + \ E\{|V_n|^s 1\{|V_n| > M\} 1\{|U_n| > \epsilon\sqrt{n}\}\} \\
&\equiv V_n + VI_n.
\end{aligned}$$

by using Wellner's moment bound (3). Thus it remains to bound VI_n. Since $E|X|^r < \infty$, we have $P(|X| \ge x) \le E|X|^r / x^r$, so

$$x^r\{F(-x) + (1 - F(x)\} \le x^r\{F(-x) + (1 - F(x-))\} \le E|X|^r \equiv K.$$

It follows that $|x|^r F(x) \le K$ and $|x|^r (1 - F(x)) \le K$, or equivalently,

$$|Q(u)| \le K_r \{u(1-u)\}^{-1/r}, \qquad 0 < u < 1.$$

Since $|V_n| \le \sqrt{n}(|Q(\xi_{(m)})| + |Q(1/2)|)$, it follows by the C_r-inequality that

$$|V_n|^s \le 2^{s-1} n^{s/2} \left\{ K_r^s [\xi_{(m)}(1 - \xi_{(m)})]^{-s/r} + |Q(1/2)|^s \right\},$$

and hence

$$\begin{aligned}
VI_n &\le 2^{s-1} n^{s/2} K_r^s E\left\{ [\xi_{(m)}(1 - \xi_{(m)})]^{-s/r} 1\{|\xi_{(m)} - 1/2| > \epsilon\} \right\} \\
&\qquad + 2^{s-1} Q(1/2)^s n^{s/2} P(|\xi_{(m)} - 1/2| > \epsilon) \\
&\equiv VI_n(a) + VI_n(b).
\end{aligned}$$

Now the second term here, $VI_n(b)$, can be handled easily using an exponential bound of Wellner (1977):

$$P(\sqrt{n}|\xi_{(m)} - p_m| \ge \sqrt{p_m q_m}\lambda) \le 2\exp(-\lambda/10)$$

where $p_m \equiv m/(n+1) = m/(2m+1)$; see Wellner (1977) and display (7) on page 454 of SW (1986) – where (7) there should read as in the last display here. This implies that

$$P(|\xi_{(m)} - 1/2| > \epsilon) \le 2\exp(-2\epsilon\sqrt{n}/5).$$

Thus we find that

$$VI_n(b) \le 2^s |Q(1/2)|^s n^{s/2} \exp(-2\epsilon\sqrt{n}/5) \to 0 \qquad \text{as} \;\; n \to \infty.$$

It remains to handle $VI_n(a)$. The best approach here is to return to the finite sample calculations and to bound much as in the proof of the pointwise convergence argument outlined below.

$$\begin{aligned}
&n^{s/2} E\left\{[\xi_{(m)}(1-\xi_{(m)})]^{-s/r} 1\{|\xi_{(m)} - 1/2| > \epsilon\}\right\} \\
&= n^{s/2} \int_{|u-1/2|>\epsilon} u^{-s/r}(1-u)^{-s/r} u^{m-1}(1-u)^m du \cdot \frac{n!}{(m-1)!(n-m)!} \\
&= n^{s/2} \int_{|u-1/2|>\epsilon} u^{m-1-s/r}(1-u)^{m-1-s/r}(1-u) du \cdot \frac{1}{\sqrt{2\pi}} 2^n (1+o(1))\sqrt{n} \\
&= n^{(s+1)/2} \frac{1}{\sqrt{2\pi}}(1+o(1)) \int_{|u-1/2|>\epsilon} 2^n u^{m-1-s/r}(1-u)^{m-1-s/r}(1-u) du \\
&= n^{(s+1)/2} \frac{1}{\sqrt{2\pi}}(1+o(1)) \int_{|u-1/2|>\epsilon} (2u)^{m-s/r}(2(1-u))^{m-1-s/r}(1-u) du \cdot 2^{-2m+2s/r} 2^n \\
&= 2^{2s/r} n^{(s+1)/2} \frac{1}{\sqrt{2\pi}}(1+o(1)) \int_{|u-1/2|>\epsilon} (4u(1-u))^{m-1-s/r}(1-u) du
\end{aligned}$$

where

$$4u(1-u) = 1 - 4(u-1/2)^2 \le \exp(-4(u-1/2)^2).$$

Therefore, for n sufficiently large,

$$\begin{aligned}
&n^{s/2}E\left\{[\xi_{(m)}(1-\xi_{(m)})]^{-s/r}1\{|\xi_{(m)}-1/2|>\epsilon\}\right\} \\
&\le 2^{2s/r}n^{(s+1)/2}\frac{2}{\sqrt{2\pi}}\int_{|u-1/2|>\epsilon}\exp(-4(u-1/2)^2(m-1-s/r))du \\
&= 2^{2s/r}n^{(s+1)/2}\frac{2}{\sqrt{2\pi}}\int_{|z|>\epsilon}\exp(-4z^2(m-1-s/r))dz \\
&= 2^{2s/r}n^{(s+1)/2}\frac{2}{\sqrt{2\pi}}\int_{|z|>\epsilon}\exp(-4z^2(m-1-s/r))dz \\
&= 2^{2s/r+1}n^{(s+1)/2}\sigma_n P(\sigma_n|Z|>\epsilon), \qquad \sigma_n^2 \equiv \frac{1}{8(m-1-s/r)}, \\
&\le 2^{2s/r+2}n^{(s+1)/2}\sigma_n\frac{1}{\epsilon/\sigma_n}\phi(\epsilon/\sigma_n) \qquad \text{by Mills ratio inequality} \\
&= 2^{2s/r+2}n^{(s+1)/2}\sigma_n^2\frac{1}{\epsilon}\frac{1}{\sqrt{2\pi}}\exp(-4\epsilon^2(m-1-s/r)) \\
&\to 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

Combining the pieces yields

$$\begin{aligned}
E|V_n|^s \le\ & M^s + M^s\left(\sup_{u:|u-1/2|\le M/\sqrt{n}}|Q'(u)|^s\right) + C_s 2^{-s/2}\left(\sup_{u:|u-1/2|\le\epsilon}|Q'(u)|^s\right) \\
&+ 2^s|Q(1/2)|^s n^{s/2}P(|\xi_{(m)}-1/2|>\epsilon) \\
&+ 2^{s-1}K_r^s 2^{2s/r+s}\frac{1}{\sqrt{2\pi}}\frac{1}{\epsilon}\frac{n^{(s+1)/2}}{8(m-1-s/r)}\exp(-4\epsilon^2(m-1-s/r)),
\end{aligned}$$

so that

$$\limsup_{n\to\infty} E|V_n|^s \le M^s\left(1+|Q'(1/2)|^s\right) + C_s 2^{-s/2}\left(\sup_{u:|u-1/2|\le\epsilon}|Q'(u)|^s\right) < \infty;$$

i.e. (2) holds.

Some of the intuition for the proof above has relied on the following development concerning convergence of the densities of $\sqrt{n}(X_{(m)}-F^{-1}(1/2))$ to $N(0,1/(4f^2(F^{-1}(1/2))))$. Note that $X_{(n/2)} = X_{(m)}$ has density

$$\frac{n!}{(m-1)!(n-m)!}F(y)^{m-1}f(y)(1-F(y))^{n-m},$$

and hence $\sqrt{n}(X_{(m)}-F^{-1}(1/2))$ has density

$$n^{-1/2}\frac{n!}{(m-1)!(n-m)!}F(F^{-1}(1/2)+xn^{-1/2})^{m-1}f(F^{-1}(1/2)+xn^{-1/2})(1-F(F^{-1}(1/2)+xn^{-1/2}))^{n-m}.$$

Thus it follows that

$$\begin{aligned}
&E\left(\sqrt{n}(X_{(m)} - F^{-1}(1/2))\right)^2 \\
&= \int_{-\infty}^{\infty} x^2 n^{-1/2} \frac{n!}{(m-1)!(n-m)!} F(F^{-1}(1/2) + xn^{-1/2})^{m-1} f(F^{-1}(1/2) + xn^{-1/2}) \\
&\quad \cdot (1 - F(F^{-1}(1/2) + xn^{-1/2}))^{n-m}\, dx \\
&= \left(\frac{1}{\sqrt{2\pi}} + o(1)\right) 2^n \int_{-\infty}^{\infty} x^2 F(F^{-1}(1/2) + xn^{-1/2})^{m-1} f(F^{-1}(1/2) + xn^{-1/2}) \\
&\qquad (1 - F(F^{-1}(1/2) + xn^{-1/2}))^{n-m}\, dx \qquad (4)
\end{aligned}$$

by Stirling's formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n, \qquad m! = (n-m)! = \sqrt{2\pi m}(m/e)^m = \sqrt{2\pi(n/2)}((n/2)/e)^{n/2},$$

and hence

$$\begin{aligned}
\frac{n!}{(m-1)!(n-m)!} &= m\frac{n!}{(m!)^2} = m\frac{\sqrt{2\pi n}(n/e)^(1 + o(1))}{\left(\sqrt{2\pi n/2}((n/2)/e)^{n/2}(1 + o(1))\right)^2} \\
&= \frac{n}{2}\frac{1}{\sqrt{2\pi}}\frac{(n/e)^n}{((n/2)/e)n} \cdot \frac{\sqrt{n}}{n/2} \cdot (1 + o(1)) \\
&= \frac{1}{\sqrt{2\pi}} 2^n \sqrt{n}(1 + o(1)).
\end{aligned}$$

Now note that

$$F(F^{-1}(1/2) + xn^{-1/2}) \to F(F^{-1}(1/2)) = 1/2$$

and, moreover,

$$g_n(x) \equiv \sqrt{n}(F(F^{-1}(1/2) + xn^{-1/2}) - 1/2) \to f(F^{-1}(1/2))x \equiv g(x)$$

for each fixed x. Thus we can rewrite the integrand of the right side of (4) as

$$\begin{aligned}
&2^n x^2 F(F^{-1}(1/2) + xn^{-1/2})^{-1} f(F^{-1}(1/2) + xn^{-1/2}) F(F^{-1}(1/2) + xn^{-1/2})^m \\
&\quad \cdot (1 - F(F^{-1}(1/2) + xn^{-1/2}))^{n-m} \\
&= x^2 F(F^{-1}(1/2) + xn^{-1/2})^{-1} f(F^{-1}(1/2) + xn^{-1/2}) \\
&\quad \cdot \{4u_n(x)(1 - u_n(x))\}^m \qquad (5)
\end{aligned}$$

where

$$u_n(x) \equiv F(F^{-1}(1/2) + xn^{-1/2}) = 1/2 + n^{-1/2} g_n(x)$$

Note that the function $r(u) \equiv 4u(1-u)$ is maximized at $u = 1/2$ with $r(1/2) = 1$ and

$$r(u) = 1 - 4(u - 1/2)^2.$$

Hence the term $\{4u_n(x)(1-u_n(x))\}^m$ can be written as

$$\begin{aligned}
\left\{1 - 4(u_n(x) - 1/2)^2\right\}^m &= \left\{1 - \frac{4g_n^2(x)}{n}\right\}^m = \left\{1 - \frac{4g_n^2(x)}{n}\right\}^{n/2} \\
&\rightarrow \exp(-4g^2(x)/2) = \exp(-2g^2(x)) \\
&= \exp\left(-\frac{x^2/2}{\frac{1/4}{f^2(F^{-1}(1/2))}}\right) = \exp\left(-\frac{x^2}{2\sigma^2}\right)
\end{aligned}$$

with $\sigma^2 \equiv (1/4)/f^2(F^{-1}(1/2))$. Since

$$\begin{aligned}
F(F^{-1}(1/2) + xn^{-1/2})^{-1} f(F^{-1}(1/2) + xn^{-1/2}) &\rightarrow F(F^{-1}(1/2))^{-1} f(F^{-1}(1/2)) \\
&= 2f(F^{-1}(1/2)) = 1/\sigma,
\end{aligned}$$

It follows that we have

$$\begin{aligned}
&E\left(\sqrt{n}(X_{(m)} - F^{-1}(1/2))\right)^2 \\
&= \left(\frac{1}{\sqrt{2\pi}} + o(1)\right) \int_{-\infty}^{\infty} x^2 F(F^{-1}(1/2) + xn^{-1/2})^{-1} f(F^{-1}(1/2) + xn^{-1/2}) \\
&\qquad \{4u_n(x)(1-u_n(x))\}^m \, dx \qquad (6) \\
&\rightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 \exp(-x^2/(2\sigma^2)) dx = \sigma^2 \qquad (7)
\end{aligned}$$

if the interchange of limit and integral can be justified. To do this we need to either find an integrable dominating function and apply the dominated convergence theorem, or use a uniform integrability argument.