Statistics 583, Problem Set 4 Solutions

Wellner; 4/25/2007

1. Suppose that F is a continuous distribution function on $R^+ = (0, \infty)$, and that $X_i \sim F_i$ where $1 - F_i(x) = (1 - F(x))^{\exp(\theta z_i)}$, $i = 1, \ldots, N$ for some numbers $z_1, \ldots, z_N$ and $\theta \in R$.
(a) Find the locally most powerful rank test of $H: \ \theta \le 0$ versus $K: \ \theta > 0$.
(b) Describe how you would cary out your test when $N = 5$ and $z_1 = z_2 = z_3 = 0$, $z_4 = z_5 = 1$.
(c) Describe how you would carry out your test when $N = 400$, $z_i = 0$ for $i = 1, \ldots, 150$, $z_i = 1$ for $i = 151, \ldots, 400$.

Solution: (a) From example 6.3.5.2 with $\Delta_i = \exp(\theta z_i)$ it follows that, with $\underline{d} = \underline{r}^{-1}$,

$$P_\theta(\underline{R} = \underline{r}) = \prod_{i=1}^{N} \frac{\exp(\theta z_{d_i})}{\sum_{j=i}^{N} \exp(\theta z_{d_j})}$$

and hence

$$\log P_\theta(\underline{R} = \underline{r}) = \sum_{i=1}^{N} \theta z_{d_i} - \log \sum_{j=i}^{N} \exp(\theta z_{d_j}) \, .$$

Therefore we compute

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\log P_\theta(\underline{R}=\underline{r})\big|_{\theta=0} &= \sum_{i=1}^N \left\{ z_{d_i} - \frac{\sum_{j=i}^N z_{d_j}}{N-i+1} \right\} \\
&= \sum_{i=1}^N z_{d_i} - \sum_{i=1}^N \sum_{j=i}^N \frac{1}{N-i+1} z_{d_j} \\
&= \sum_{i=1}^N z_{d_i} - \sum_{j=1}^N \left(\sum_{i=1}^N 1_{i\le j} \frac{1}{N-i+1} \right) z_{d_j} \\
&= \sum_{i=1}^N z_{d_i} - \sum_{i=1}^N \left(\sum_{j=1}^N 1_{j\le i} \frac{1}{N-j+1} \right) z_{d_i} \\
&= \sum_{i=1}^N z_{d_i} \left(1 - \sum_{j=1}^N 1_{j\le i} \frac{1}{N-j+1} \right) \\
&= \sum_{i=1}^N z_{d_i} \left(1 - \sum_{j=1}^i \frac{1}{N-j+1} \right) \\
&\equiv \sum_{i=1}^N z_{d_i} a_N(i) \\
&= \sum_{i=1}^N z_i a_N(r_i) \equiv S_N(\underline{r})
\end{aligned}
$$

where

$$
\begin{aligned}
a_N(i) &= 1 - \sum_{j=1}^i \frac{1}{N-j+1} = 1 - \sum_{N-i+1}^N \frac{1}{k} \\
&\doteq 1 + \log(1 - \frac{i}{N+1}) \\
&= -\left(-\log(1 - \frac{i}{N+1}) - 1 \right).
\end{aligned}
$$

Thus the test of $H: \ \theta \le 0$ versus $K: \ \ \theta > 0$ is "reject H if $S_N(\underline{R}) > k_\alpha$" where k_α is determined so that $P_0(S_N(\underline{R}) > k_\alpha) = \alpha$.

(b) When $N = 5$, $m = 3$, $n = 2$, and $z_1 = z_2 = z_3 = 0$, $z_4 = z_5 = 1$, then the test becomes "reject H if $S_N(\underline{R}) > k_\alpha$ where now

$$
S_N(\underline{r}) = \sum_{i=4}^5 a_5(r_i) = \sum_{j=1}^2 a_5(q_j)
$$

where the $a_5(i)$ are given by

$$(a_5(1), a_5(2), a_5(3), a_5(4), a_5(5)) = (.8, .55, .2166\ldots, -.2833\ldots, -1.2833\ldots).$$

Equivalently, we reject if $\widetilde{S}_N < \widetilde{k}_\alpha$ where

$$\widetilde{S}_N(\underline{r}) = \sum_{i=4}^{5} b_N(r_i) = \sum_{j=1}^{2} b_N(q_j)$$

and

$$b_N(i) \equiv \sum_{k=N-i+1}^{N} \frac{1}{k}.$$

Here $(b_5(1), b_5(2), b_5(3), b_5(4), b_5(5)) = (.2, .45, .7833, 1.2833, 2.2833)$. The null distribution of the test statistic $\widetilde{S}_5$ has equal mass .10 on each of the $\binom{5}{3} = 10$ points corresponding to observing

$$\underline{q} = (q_1, q_2) = (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5),$$

namely

$$\tilde{S}_5(q) = .65, .9833, 1.4833, 2.4833, 1.233, 1.733, 2.733, 2.066, 3.0666, 3.5666;$$

ordering these gives the following table:

Table 1:

$\underline{q}$	(1,2)	(1,3)	(2,3)	(1,4)	(2,4)	(3,4)	(1,5)	(2,5)	(3,5)	(4,5)
$S_5(\underline{q})$	.65	.9833	1.233	1.4833	1.733	2.066	2.4833	2.733	3.0666	3.5666

We reject H if $\widetilde{S}_5$ is "too small" relative to this distribution. Thus if $\underline{Q} = (3,5)$ is observed and $\alpha = 1/5$, then we would reject H.

(c) When $N = 400$, $m = 150$, $n = 250$, then it's quite reasonable to use the approximate scores $\widetilde{b}_N(i) \equiv -\log(1 - i/(N+1))$ and the test becomes "reject H if $\overline{B}_n \equiv n^{-1}\sum_{j=1}^{n} b_N(Q_j) < k_\alpha$" where k_α is chosen so that

$$P_0(\overline{B}_n < k_\alpha) = \alpha \qquad \text{approximately.}$$

But here the null distribution of $\overline{B}_n$ is the same as the distributon of the sample mean in sampling without replacement from an urn containing balls with the numbers $b_N(1), \ldots, b_N(N)$ written on the balls. Since $\overline{b}_N \to 1$ and $nVar(\overline{B}_n) \to \lambda$, by problem 4.3(ii) and since the Noether condition holds by problem 4.3(iii), it follows from the WWHN finite sampling CLT that

$$\sqrt{n}(\overline{B}_n - 1) \to_d N(0, \lambda)$$

as $n \to \infty$. Thus the test becomes "reject H if $\sqrt{n}(\overline{B}_n - 1)/\sqrt{m/N} < -z_\alpha$" where $P(Z < -z_\alpha) = \Phi(-z_\alpha) = \alpha$.

2. Let $U_{m,n} \equiv T(\mathbb{F}_m, \mathbb{G}_n)$ where $T(F, G) = \int F dG = P(X \le Y)$ is the Mann-Whitney functional and $\mathbb{F}_m$ and $\mathbb{G}_n$ are the empirical df's of $X_1, \ldots, X_m$ i.i.d. with df F, $Y_1, \ldots, Y_n$ i.i.d. with df G.
(a) Show that

$$mnU_{m,n} + n(n+1)/2 = W_{m,n} \equiv \sum_{j=1}^{n} Q_j = \sum_{j=1}^{n} R_{m+j}\,.$$

(b) Show that $EU_{m,n} = P(X \le Y) = \int F dG$ and that

$$\begin{aligned} &Var(\sqrt{mn}U_{m,n}) \\ &= (n-1)\int (1-G)^2 dF + (m-1)\int F^2 dG - \; (N-1)(\int F dG)^2 + \int F dG \\ &= (n-1)Var[1-G(X)] + (m-1)Var[F(Y)] + \int F dG(1 - \int F dG)\,. \end{aligned}$$

(c) When $F = G$ use the results of A and B to compute $E_{(F,F)}W_{m,n}$ and $Var_{(F,F)}(W_{m,n})$. (This should agree with calculations for the Wilcoxon rank sum form of the statistic under the null hypothesis via finite sampling calculations.)

Solution: (a) Using empirical distribution function notation, $N\mathbb{H}_N = m\mathbb{F}_m + n\mathbb{G}_n$, so

$$\begin{aligned} mnU_{m,n} &= \int m\mathbb{F}_m d(n\mathbb{G}_n) = \int N\mathbb{H}_N d(n\mathbb{G}_n) - \int n\mathbb{G}_n d(n\mathbb{G}_n) \\ &= \sum_{j=1}^{n} N\mathbb{H}_N(Y_j) - \sum_{j=1}^{n} n\mathbb{G}_n(Y_{(j)}) \\ &= \sum_{j=1}^{n} R_{m+j} - \sum_{j=1}^{n} j \\ &= \sum_{j=1}^{n} R_{m+j} - n(n+1)/2\,. \end{aligned}$$

(b) The expectation is easy:

$$E(U_{m,n}) = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n} P(X_i \le Y_j) = P(X_1 \le Y_1) = \int F dG\,.$$

For the variance, we first calculate

$$\begin{aligned}
E[mnU_{m,n}]^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{n} E1_{[X_i\le Y_j, X_k \le Y_l]} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} E1_{[X_i \le Y_j]} + \sum_{i\neq k}\sum_{j=1}^{n} P(X_i \le Y_j, X_k \le Y_j) \\
&\qquad + \sum_{i=1}^{m}\sum_{j\neq l} P(X_i \le Y_j, X_i \le Y_l) + \sum_{i\neq k}\sum_{j \neq l} P(X_i \le Y_j, X_k \le Y_l) \\
&= mnP(X_1 \le Y_1) + m(m-1)nP(X_1 \le Y_1, X_2 \le Y_1) \\
&\qquad + mn(n-1)P(X_1 \le Y_1, X_1 \le Y_2) \\
&\qquad + m(m-1)n(n-1)P(X_1 \le Y_1, X_2 \le Y_2)\,,
\end{aligned}$$

$$P(X_1 \le Y_1) = \int F dG,$$

$$P(X_1 \le Y_1, X_2 \le Y_1) = EP(X_1 \le Y_1, X_2 \le Y_1 | Y_1) = \int F^2(x) dG(x)\,,$$

$$P(X_1 \le Y_1, X_1 \le Y_2) = EP(X_1 \le Y_1, X_1 \le Y_2 | X_1) = \int (1 - G(x-))^2 dF(x),$$

and

$$P(X_1 \le Y_1, X_2 \le Y_2) = P(X_1 \le Y_1)^2 = (\int F dG)^2\,.$$

It follows by algebra that

$$\begin{aligned}
\mathrm{Var}(mnU_{m,n}) &= E(mnU_{m,n})^2 - \{E(mnU_{m,n})\}^2 \\
&= mn\int FdG + m(m-1)n\int F^2dG \\
&\qquad + mn(n-1)\int(1-G(x-))^2dF(x) \\
&\qquad + m(m-1)n(n-1)\{\int FdG\}^2 - (mn\int FdG)^2 \\
&= m(m-1)n\{\int F^2dG - (\int FdG)^2\} \\
&\qquad + mn(n-1)\{\int(1-G(x-))^2dF(x) - (\int FdG)^2\} \\
&\qquad - mn\int FdG(1-\int FdG)\,.
\end{aligned}$$

By noting that

$$\begin{aligned}
\int FdG = P(X \le Y) &= 1 - P(X > Y) \\
&= 1 - \int G(x-)dF(x) = \int(1-G(x-))dF(x)\,,
\end{aligned}$$

this yields the claimed variance formula (to within a left limit):

$$\begin{aligned}
\mathrm{Var}(\sqrt{mn}U_{m,n}) &= (m-1)Var(F(Y)) + (n-1)Var(1-G(X-)) \\
&\qquad + \int FdG\left(1-\int FdG\right)\,.
\end{aligned}$$

(c) When $F = G$ continuous we find that

$$E(U_{mn}) = \int FdF = 1/2\,,$$

and, since now $Var[F(Y)] = Var[G(X)] = 1/12$,

$$\begin{aligned}
\mathrm{Var}(\sqrt{mn}U_{m,n}) &= (m-1)\frac{1}{12} + (n-1)\frac{1}{12} + \frac{1}{4} \\
&= (N-2)\frac{1}{12} + \frac{1}{4} = (N+1)\frac{1}{12}\,.
\end{aligned}$$

Hence from part (a) it follows that

$$E(\sum_{j=1}^{n} Q_j) = n(n+1)/2 + mnE(U_{m,n}) = n(N+1)/2$$

and

$$\text{Var}(\sum_{j=1}^{n} Q_j) = mn\text{Var}(\sqrt{mn}U_{m,n}) = mn(N+1)\frac{1}{12}$$

both of which agree with the finite sampling calculations of problem 2.(a) of problem set 3.

3. Suppose that $\mathcal{F}_+$ is the class of distribution functions F on $\mathbb{R}^+$, and consider the functional $T(F)$ defined for a fixed $x_0 \in R^+$ by

$$T(F) \equiv e_F(x_0) \equiv E_F(X - x_0 | X > x_0) = \frac{\int_{x_0}^{\infty}(1 - F(t))dt}{1 - F(x_0)} .$$

This functional is the *mean residual life* functional.
(a) For what collection of df's F_0 is T weakly continuous at F_0? For what collection of df's F_0 is T continuous at F_0 with respect to the Kolmogorov metric?
(b) Find the influence function of $T(F)$.

Solution: (a) Suppose that $F_n \to_d F$, and write

$$T(F) = \frac{\int_0^{\infty}(x - x_0)1_{(x_0,\infty)}(x)dF(x)}{1 - F(x)}.$$

Thus

$$\begin{aligned} T(F_n) &= \frac{\int_0^{\infty}(x - x_0)1_{(x_0,\infty)}(x)dF_n(x)}{1 - F_n(x)} && (1) \\ &\to \frac{\int_0^{\infty}(x - x_0)1_{(x_0,\infty)}(x)dF(x)}{1 - F(x)} = T(F) && (2) \end{aligned}$$

by the Helly-Bray lemma if x_0 is a continuity point of F and if x is uniformly integrable with respect to the sequence F_n:

$$\limsup_n E_{F_n}\{X 1_{[X \ge \lambda]}\} \to 0$$

as $\lambda \to \infty$. If $F_n \to F$ with respect to the Kolmogorov metric d_K, then $F_n \to F$, so the previous argument goes through under the same assumptions, but (2) may continue to hold even when F is discontinuous at x_0. To see this, note that $d_K(F_n, F) \to 0$ implies that $F_n(x_0) \to F(x_0)$ (even if F is discontinuous at x_0), while the numerator of $T(F_n)$ can be written as

$$\begin{aligned} \int_{x_0}^{\infty}(1 - F_n(t))dt &= \int_{x_0}^{M}(1 - F_n(t))dt + \int_{M}^{\infty}(1 - F_n(t))dt \\ &\to \int_{x_0}^{M}(1 - F(t))dt + \epsilon = \int_{x_0}^{\infty}(1 - F(t))dt + 2\epsilon \end{aligned}$$

by using $d_K(F_n, F) \to 0$ and uniform integrability.

(b) First note that with $F_t \equiv (1-t)F + tG$ we have both

$$\frac{d}{dt}(1 - F_t(x_0))\Big|_{t=0} = -(G-F)(x_0)$$

and

$$\frac{d}{dt}\int_{x_0}^{\infty}(1 - F_t(y))dy\Big|_{t=0} = -\int_{x_0}^{\infty}(G-F)(y)dy\,.$$

Thus by the product rule we calculate

$$\begin{aligned}
\frac{d}{dt}T(F_t)\Big|_{t=0} &= -\frac{\int_{x_0}^{\infty}(G-F)(y)dy}{1-F(x_0)} + \frac{\int_{x_0}^{\infty}(1-F(y))dy}{(1-F(x_0))^2}(G-F)(x_0) \\
&= e_F(x_0)\frac{(G-F)(x_0)}{1-F(x_0)} - \frac{\int_{x_0}^{\infty}(G-F)(y)dy}{1-F(x_0)}\,.
\end{aligned}$$

Taking $G = \delta_x = 1_{[x,\infty)}$ yields the influence function for T at F:

$$\begin{aligned}
IC(x;T,F) &= e_F(x_0)\frac{(1_{[x,\infty)}(x_0) - F(x_0))}{1-F(x_0)} - \frac{\int_{x_0}^{\infty}(1_{[x,\infty)}(y) - F(y))dy}{1-F(x_0)} \\
&= e_F(x_0)\frac{(1_{[0,x_0]}(x) - F(x_0))}{1-F(x_0)} - \frac{\int_{x_0}^{\infty}(1_{[0,y]}(x) - F(y))dy}{1-F(x_0)} \\
&= \begin{cases} e_F(x_0) - \frac{\int_{x_0}^{\infty} 1-F(y)dy}{1-F(x_0)} & x \le x_0 \\ \frac{-F(x_0)}{1-F(x_0)}e_F(x_0) - \frac{\int_{x_0}^{\infty} 1_{[0,y]}(x_0)-F(y)dy}{1-F(x_0)} & x > x_0 \end{cases} \\
&= \begin{cases} 0 & x \le x_0 \\ \frac{-F(x_0)}{1-F(x_0)}e_F(x_0) - \frac{\int_{x_0}^{x} -F(y)dy}{1-F(x_0)} - \frac{\int_{x}^{\infty} 1_{[0,y]}(x_0)-F(y)dy}{1-F(x_0)} & x > x_0 \end{cases} \\
&= \begin{cases} 0 & x \le x_0 \\ \frac{(x-x_0)-e_F(x_0)}{1-F(x_0)} & x > x_0 \end{cases} \\
&= \frac{[(x-x_0) - e_F(x_0)]1_{(x_0,\infty)(x)}}{1-F(x_0)}\,.
\end{aligned}$$

Note that

$$E_F[IC^2(X;T,F)] = \frac{Var(X - x_0|X > x_0)}{1-F(x_0)}\,.$$

4. Let F be a distribution function on $\mathbb{R}^2$ with finite second moments, and let $\rho(F)$ be the correlation coefficient

$$\rho(F) = \frac{Cov_F(X,Y)}{\sqrt{Var_F(X)Var_F(Y)}} .$$

Assume that $|\rho(F)| < 1$.
(a) Give an example of a sequence of bivariate distributions $\{F_n\}$ satisfying $F_n \to F$, but $\rho(F_n) \to 1$.
(b) Find a collection $\mathcal{F}$ of distribution functions on $\mathbb{R}^2$ so that ρ is weakly continuous on $\mathcal{F}$.

Solution: (a) Without loss of generality we may suppose that F is a bivariate distribution function with zero means, $E_F(X) = E_F(Y) = 0$. Let $F_n = (1 - n^{-1})F + n^{-1}\delta_{(a_n,b_n)}$ with $(a_n, b_n) \in \mathbb{R}^2$. Note that F_n has marginal distribution functions $F_{n,X} = (1-n^{-1})F_X + n^{-1}\delta_{a_n}$, $F_{n,Y} = (1-n^{-1})F_Y + n^{-1}\delta_{b_n}$ respectively where F_X and F_Y are the marginal df's of F. Thus we compute

$$\begin{aligned}
Cov_{F_n}(X,Y) &= E_{F_n}(XY) - E_{F_n}(X)E_{F_n}Y \\
&= (1 - n^{-1}Cov_F(X,Y) + n^{-1}a_n b_n - (n^{-1}a_n)(n^{-1}b_n) \\
&= (1 - n^{-1}Cov_F(X,Y) + n^{-1}(1-n^{-1})a_n b_n, \\
Var_{F_n}(X) &= E_{F_n}(X^2) - (E_{F_n}(X))^2 = (1-n^{-1})Var_F(X) + n^{-1}(1-n^{-1})a_n^2, \\
Var_{F_n}(Y) &= E_{F_n}(Y^2) - (E_{F_n}(Y))^2 = (1-n^{-1})Var_F(Y) + n^{-1}(1-n^{-1})b_n^2.
\end{aligned}$$

Choosing $a_n = b_n = n$ yields

$$\begin{aligned}
Cov_{F_n}(X,Y) &= n + o(n) = n(1+o(1)), \\
Var_{F_n}(X) &= n + o(n) = n(1+o(1)), \\
Var_{F_n}(Y) &= n + o(n) = n(1+o(1)).
\end{aligned}$$

Thus we find that

$$\rho(F_n) = \frac{Cov_{F_n}(X,Y)}{\sqrt{Var_{F_n}(X)Var_{F_n}(Y)}} = \frac{n(1+o(1))}{n(1+o(1))} \to 1$$

as $n \to \infty$. Thus ρ is weakly discontinuous at every F.

(b) Consider the following collection of distributions on R^2: for some $r > 2$ and $M < \infty$

$$\mathcal{F}_{r,M} \equiv \{F : \quad E_F|X|^r \le M, \; E_F|Y|^r \le M\} .$$

Then ρ is weakly-continuous on $\mathcal{F}_{r,M}$ at any F with $Var_F(X) > 0$ and $Var_F(Y) > 0$. Here is a proof: let $\{F_n\} \subset \mathcal{F}_{r,M}$ satisfy $F_n \to_d F$. Then with $(X_n, Y_n) \sim F_n$ and $(X,Y) \sim F$ we have $(X_n, Y_n) \to_d (X,Y)$, and by a Skorokhod construction there exist $(X_n^*, Y_n^*) =_d (X_n, Y_n)$ and $(X^*, Y^*) =_d (X, Y)$ defined on a common probability space and satisfying $(X_n^*, Y_n^*) \rightarrow_{a.s.} (X^*, Y^*)$. But because $\{F_n \subset \mathcal{F}_{r,M}$, X_n^2, Y_n^2, and $|X_n Y_n|$ are all uniformly integrable: since $r > 2$,

$$EX_n^2 1_{[X_n^2 \geq \lambda]} \;\; \leq \;\; \frac{1}{\lambda^{r-2}} E|X_n|^r \leq \frac{M}{\lambda^{r-2}}$$

so

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} EX_n^2 1_{[X_n^2 \geq \lambda]} \;\; \leq \;\; \lim_{\lambda\to\infty} \frac{M}{\lambda^{r-2}} = 0$$

and similarly for $\{Y_n^2\}$, so the uniform integrability of $|X_n Y_n|$ follows by Cauchy-Schwarz. The same holds true for the (X_n^*, Y_n^*) pairs since the uniform integrability only depends on the (marginal) distributions. Thus by Vitali's theorem it follows that

$$EX_n^s = EX_n^* s \rightarrow EX^* s = EX^s$$

and

$$EY_n^s = EY_n^* s \rightarrow EY^* s = EY^s$$

for $s = 1, 2$, while Vitali also yields

$$EX_n Y_n = EX_n^* Y_n^* \rightarrow EX^* Y^* = EXY \, .$$

Therefore

$$Var_{F_n}(X_n) \rightarrow Var_F(X), Var_{F_n}(Y_n) \rightarrow Var_F(Y), \tag{3}$$

and

$$Cov_{F_n}(X_n, Y_n) \rightarrow Cov_F(X, Y). \tag{4}$$

Since we have assumed that $Var_F(X) > 0$ and $Var_F(Y) > 0$, (3) and (4) yield

$$\rho(F_n) = \frac{Cov_{F_n}(X_n, Y_n)}{\sqrt{Var_{F_n}(X_n) \cdot Var_{F_n}(Y_n)}} \rightarrow \frac{Cov_F(X, Y)}{\sqrt{Var_F(X) \cdot Var_F(Y)}} = \rho(F);$$

i.e. ρ is continuous on $\mathcal{F}_{r,M}$ at any F with positive variances.

It is interesting to note that the hypothesis $\{F_n\} \subset \mathcal{F}_{r,M}$ *cannot be weakened to* $\{F_n\} \subset \mathcal{F}_{2,M}$ (and hence it can also not be weakened to the still larger class $\mathcal{F}_{2,\infty}$). Here is a counterexample. Let F be a d.f. on R^2 with $EX = 0 = EY$ and $EX^2 = 1 = EY^2$, and $\rho(F) < 1$ where $(X, Y) \sim F$. Let $M > 1$ be a big number, and consider the class

$$\mathcal{F}_{2,M} = \{F \text{ on } R^2 : \; E_F X^2 \leq M, E_F Y^2 \leq M\} \, .$$

Let $a_n, b_n > 0$; we will specify them in terms of M shortly. Consider the sequence of d.f.'s $\{F_n\} \subset \mathcal{F}_{2,M}$ defined by

$$F_n(x,y) = (1 - \frac{1}{n})F(x,y) + \frac{1}{2n}\delta_{(a_n,b_n)} + \frac{1}{2n}\delta_{(-a_n,-b_n)}.$$

Then for any bounded and continuous function $\psi : R^2 \to R$,

$$\begin{aligned} \int \psi dF_n &= (1 - \frac{1}{n}) \int \psi dF + \frac{1}{2n}\psi(a_n, b_n) + \frac{1}{2n}\psi(-a_n, -b_n) \\ &\to \int \psi dF, \end{aligned}$$

so $F_n \to_d F$. Furthermore, with $(X_n, Y_n) \sim F_n$,

$$EX_n = (1 - 1/n)EX = 0, EY_n = 0,$$

$$EX_n^2 = = (1 - 1/n)EX^2 + \frac{a_n^2}{n} = (1 - 1/n) + \frac{a_n^2}{n} = M$$

if $a_n^2 = n\{M - (1 - 1/n)\}$. Similarly,

$$EY_n^2 = (1 - 1/n) + \frac{b_n^2}{n} = M$$

if $b_n^2 = n\{M - (1 - 1/n)\}$. With these choices of a_n and b_n,

$$Cov(X_n, Y_n) = (1 - 1/n)Cov(X, Y) + \frac{a_n b_n}{n},$$

$$\begin{aligned} \rho(F_n) &= \frac{Cov(X_n, Y_n)}{Var(X_n)Var(Y_n)} \\ &= \frac{(1 - 1/n)Cov(X, Y) + M - (1 - 1/n)}{\sqrt{M^2}} \\ &\to \frac{\rho(F) + M - 1}{M} \neq \rho(F). \end{aligned}$$

Thus $\rho(F)$ is not continuous on $\mathcal{F}_{2,M}$.

5. Consider the collection $\mathcal{F}_0$ of distribution functions F on R^+ with $0 < E_F X < \infty$ and $E_F X^2 < \infty$. Let $T(F) \equiv \sigma(F)/\mu(F)$ for $F \in \mathcal{F}_0$ where $\sigma^2(F) = Var_F(X)$ and $\mu(F) = E_F(X)$. This is the *coefficient of variation of F*. Find the influence function of $T(F)$.

Solution: $T(F) = \sqrt{\sigma^2(F)}/\mu(F)$ where we already know that

$$\dot{\mu}(F; G-F) = \frac{d}{dt}\mu(F_t)|_{t=0} = \int (x - \int xdF)dG(x)$$

and

$$\dot{\sigma}^2(F; G-F) = \int \{(x-\mu_F)^2 - \sigma_F^2\}dG(x)\,.$$

Thus, by the product rule for differentiation,

$$\dot{T}(F; G-F) \quad = \quad \frac{1}{2}\frac{\sigma(F)}{\mu(F)}\frac{1}{\sigma^2(F)}\dot{\sigma}^2(F; G-F) - \frac{\sigma^2(F)}{\mu^2(F)}\frac{1}{\sigma(F)}\dot{\mu}(F; G-F),$$

and

$$IC(x; T, F) = \frac{1}{2}T(F)\{\left(\frac{x-\mu_F}{\sigma_F}\right)^2 - 1\} - T^2(F)\left(\frac{x-\mu_F}{\sigma_F}\right)\,.$$