

## Statistics 583, Midterm Exam

Wellner; 5/4/2007

**Instructions:** This is an “in class” and “closed-book” exam. Please do it completely on your own with no books or notes.

1. (30 points) **Define** any *three* of the following terms.
  - (a) A maximal invariant with respect to a group  $G$  of transformations  $g$  on the sample space  $\mathcal{X}$ .
  - (b) A  $G$ -invariant test function  $\phi$  (with respect to a group  $G$ ).
  - (c) The Lévy metric  $d_L$  on the set of distribution functions  $\mathcal{F}$  on  $\mathbb{R}$ .
  - (d) The Prohorov metric  $d_{Pr}$  on the set of all probability measures on a metric space  $(M, d)$  with the Borel  $\sigma$ -field.
  - (e) A Hadamard differentiable functional  $T : \mathcal{F} \rightarrow \mathbb{R}$  with respect to the Kolmogorov metric  $\|\cdot\|_\infty$ .
2. (30 points) Give a complete **statement** of any *two* of the following results:
  - (a) Hoeffding’s formula for the distribution of ranks under the alternative.
  - (b) The Wald- Wolfowitz-Noether-Hajek finite sampling central limit theorem.
  - (c) Varadarajan’s theorem concerning weak convergence of the empirical measure  $\mathbb{P}_n$  when  $X_1, \dots, X_n$  are i.i.d.  $P$  on a metric space  $(M, d)$ .
  - (d) A theorem about the existence of a UMP  $G$ -invariant test in the case that both the  $G$ -maximal invariant and the  $\overline{G}$ -maximal invariant are real-valued.

**Do both problem 3 and problem 4.**

3. (40 points) Consider the collection of functionals

$$T(F) \equiv T_x(F) = P_F(x \in [X_1 \wedge X_2, X_1 \vee X_2))$$

where  $X_1, X_2 \sim F$  are independent (and  $X_1 \wedge X_2 = \min\{X_1, X_2\}$ ,  $X_1 \vee X_2 = \max\{X_1, X_2\}$ ). This is the simplicial depth function we discussed in Example 7.1.12 in the case  $d = 1$ .

- (a) For fixed  $x \in \mathbb{R}$ , express this functional explicitly in terms of  $F$  and show that  $T_x(F) \leq 1/2$  for all  $x \in \mathbb{R}$ . What value of  $x$  maximizes  $T_x(F)$ ?
- (b) Under what conditions is  $T_x(F)$  weakly continuous? Under what conditions is  $T_x(F)$  continuous with respect to the Kolmogorov metric  $d_K(F, G) = \|F - G\|_\infty$ ?
- (c) Find the Gateaux derivative of  $T_x(F)$  for fixed  $x$  and thereby find the influence function  $\psi_{x,F}(y)$  as a function of  $y$  for each fixed  $x$ . Be careful about confusing  $x$  and  $y$  here!
- (d) Using any method you want, show that  $\sqrt{n}(T_x(\mathbb{F}_n) - T_x(F)) \rightarrow_d N(0, V_x^2)$  and find the asymptotic variance  $V_x^2$  in terms of  $F$  and  $x$ .
- (e) What can you say about the stochastic process  $R_n(x) = \sqrt{n}(T_x(\mathbb{F}_n) - T_x(F))$ ,  $x \in \mathbb{R}$ ?

4. (36 points). Suppose that under the null hypothesis  $H_c$ ,  $X_1, \dots, X_N$  are i.i.d.  $F \in \mathcal{F}_c$ , while under the alternative hypothesis  $K_1$ , the random variables  $X_1, \dots, X_N$  have joint density  $h$  given by

$$h(\underline{x}) = \prod_{i=1}^N \lambda_i \exp(-\lambda_i x_i) 1_{(0, \infty)}(x_i)$$

where  $\lambda_i = \mu e^{\nu c_i}$  for some fixed numbers (covariates)  $c_1, \dots, c_N$ , and constants  $\mu > 0$  and  $\nu \in R$ .

- (a) Find a most powerful similar test of size  $\alpha$  for testing  $H_c$  versus  $K_1$ . Does your test depend on the values of the constants  $\mu$  and/or  $\nu$ ?
- (b) If  $N = 3$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 2$ , and  $\lambda_3 = 3$ , and we observe  $(X_1, X_2, X_3) = (4, 2, 1)$ , carry out the test in (a) at level  $\alpha = 1/6$ .
- (c) Find a locally most powerful rank test of  $H_0 : \nu = 0$  versus  $H_1 : \nu > 0$ . Does it matter that  $\mu$  is unknown in both  $H_0$  and  $H_1$ ?