Statistics 583, Problem Set 2 Solutions

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1. Let X_{ij} for $j = 1, \ldots, n_i$ and $i = 1, \ldots, s$ be independent, normally distributed random variables with common variance σ^2, and suppose that $EX_{ij} = \mu_i$. Consider testing $H: \ \mu_1 = \mu_2 = \cdots = \mu_s$ versus $K: \ \mu_i \neq \mu_j$ for some $i \neq j$.
(a) Put this in the form of the general linear model and find the UMP invariant test of H versus K.
(b) What is the distribution of the test statistic under the general hypothesis? [See Lehmann and Romano, section 7.3, pages 285 - 286, and Ferguson, example 5.9.1, page 265.]
(c) Suppose that $s = 4$ and $n_i = 6$ for $i = 1, \ldots, 4$. Suppose further that that observed data yield $\overline{X}_{1\cdot} = 0$, $\overline{X}_{2\cdot} = 4$, $\overline{X}_{3\cdot} = 5$, $\overline{X}_{4\cdot} = 7$ and $\hat{\sigma}^2 = 10$. Compute the F-statistic derived in (a) and the $p-$value for this data for testing H versus K.

Solution: (a) Here $EX_{ij} = \mu_i$, for $j = 1, \ldots, n_i$, $j = 1, \ldots, s$. Note that

$$\sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - \mu_i)^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - X_{i\cdot})^2 + \sum_{i=1}^{s} n_i (X_{i\cdot} - \mu_i)^2.$$

Hence, it follows that the estimates of μ_i under the full model are

$$\hat{\mu}_i = X_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, \qquad i = 1, \ldots, s$$

and

$$\min_{\mu} \sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - \mu_i)^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - X_{i\cdot})^2.$$

Under the hypothesis $H: \mu_1 = \cdots = \mu_s$ we write

$$\sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - \mu_i)^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - X_{\cdot\cdot})^2 + n(X_{\cdot\cdot} - \mu)^2.$$

where $n \equiv n_1 + \cdots + n_s$. hence

$$\hat{\hat{\mu}}_i = \hat{\hat{\mu}} = \frac{\sum_{i=1}^{s} \sum_{j=1}^{n_s} X_{ij}}{\sum_{i=1}^{s} n_i} = \frac{\sum_{i=1}^{s} n_i \hat{\mu}_i}{n} \equiv X_{\cdot\cdot}$$

and

$$\min_{\mu \in H} \sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - \mu_i)^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - X_{\cdot\cdot})^2.$$

Here $k = s$, $r = s - 1$. (b) Thus the F statistic for testing the null hypothesis H versus $K: \ \mu_i \neq \mu_j$ for some $i \neq j$

$$\begin{aligned} F &= \frac{\sum_{i=1}^{s} \sum_{j=1}^{n_i} (\hat{\mu}_i - \hat{\hat{\mu}}_i)^2/(s-1)}{\sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - X_{\cdot\cdot})^2/(n-s)} \\ &= \frac{\sum_{i=1}^{s} n_i (X_{i\cdot} - X_{\cdot\cdot})^2/(s-1)}{\sum_{i=1}^{s} \sum_{j=1}^{n_i} (X_{ij} - X_{\cdot\cdot})^2/(n-s)} \\ &\sim F_{s-1, n-s}(\delta^2) \end{aligned}$$

where the non-centrality parameter is

$$\delta^2 = \frac{\sum_{i=1}^{s} n_i (\mu_i - \sum_{i=1}^{s} n_i \mu_i / n)^2}{\sigma^2}.$$

(c) When $s = 4$ and $n_i = 6$ for $i = 1, \ldots, 4$, and we observe data leading to $\overline{X}_{1\cdot} = 0$, $\overline{X}_{2\cdot} = 4$, $\overline{X}_{3\cdot} = 5$, $\overline{X}_{4\cdot} = 7$, and $\hat{\sigma}^2 = 10$, then $X_{\cdot\cdot} = 4$, and the value of F statistic is

$$F = \frac{[6(0-4)^2 + 6(4-4)^2 + 6(5-4)^2 + 6(7-4)^2]/(4-1)}{10} = 5.2.$$

The resulting p-value is $P(F_{3,20} \geq 40.6) = .0081$.

2. Let X_i, $i = 1, \ldots, I$ and $Y_j = 1, \ldots, J$ satisfy the general linear hypothesis with $EX_i = \alpha_1 + \beta_1 u_i$ and $EY_j = \alpha_2 + \beta_2 v_j$, where the u_i and v_j are known and $\sum u_i = 0 = \sum v_j$, $\sum u_i^2 = I$ and $\sum v_j^2 = J$.
(a) Find the UMP invariant test of $H_0 : \beta_1 = \beta_2$ versus $K_0 : \beta_1 \neq \beta_2$.
(b) Find the UMP invariant test of H_0': $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Solution: (a) Here $n = I + J$, $k = 4$, and $r = 1$. It is easily seen that

$$\hat{\alpha}_1 = \overline{X}, \qquad \hat{\beta}_1 = \overline{uX} \equiv \frac{1}{n} \sum_{i=1}^{I} u_i X_i;$$

and

$$\hat{\alpha}_2 = \overline{Y}, \qquad \hat{\beta}_2 = \overline{vY} \equiv \frac{1}{J} \sum_{j=1}^{J} v_j Y_j.$$

Under $H: \beta_1 = \beta_2$, we find that $\hat{\hat{\alpha}}_1 = \overline{X}$, $\hat{\hat{\alpha}}_2 = \overline{Y}$, and

$$\hat{\hat{\beta}}_1 = \hat{\hat{\beta}}_2 = \frac{\sum_{i=1}^I u_i X_i + \sum_{j=1}^J v_j Y_j}{I+J}.$$

Hence the $F-$ statistic for testing H versus $K: \beta_1 \neq \beta_2$ is

$$\begin{aligned} F &= \frac{\sum_{i=1}^I \hat{\alpha}_1 + \hat{\beta}_1 u_i - \hat{\hat{\alpha}}_1 - \hat{\hat{\beta}}_1 u_i)^2 + \sum_{j=1}^J \hat{\alpha}_2 + \hat{\beta}_2 v_j - \hat{\hat{\alpha}}_2 - \hat{\hat{\beta}}_2 v_j)^2}{[\sum_{i=1}^I (X_i - \hat{\alpha}_1 - \hat{\beta}_1 u_i)^2 + \sum_{j=1}^J (Y_j - \hat{\alpha}_2 - \hat{\beta}_2 v_j)^2]/(I+J-4)} \\ &= \frac{\frac{IJ}{I+J}(\hat{\beta}_1 - \hat{\beta}_2)^2}{[\sum_{i=1}^I (X_i - \hat{\alpha}_1 - \hat{\beta}_1 u_i)^2 + \sum_{j=1}^J (Y_j - \hat{\alpha}_2 - \hat{\beta}_2 v_j)^2]/(I+J-4)} \end{aligned}$$

and the test becomes "reject H if $F > F_{1,I+J-4,\alpha}$."
(b). In this case $n = I + J$, $k = 4$, and $r = 2$. The estimators are the same under the "big model" as in part (a). Under the hypothesis H we find that

$$\hat{\hat{\alpha}}_1 = \hat{\hat{\alpha}}_2 = (I\overline{X} + J\overline{Y})/(I+J).$$

and, as in (a),

$$\hat{\hat{\beta}}_1 = \hat{\hat{\beta}}_2 = \frac{\sum_{i=1}^I u_i X_i + \sum_{j=1}^J v_j Y_j}{I+J}.$$

Thus the $F-$ statistic becomes

$$\begin{aligned} F &= \frac{[\sum_{i=1}^I \hat{\alpha}_1 + \hat{\beta}_1 u_i - \hat{\hat{\alpha}}_1 - \hat{\hat{\beta}}_1 u_i)^2 + \sum_{j=1}^J \hat{\alpha}_2 + \hat{\beta}_2 v_j - \hat{\hat{\alpha}}_2 - \hat{\hat{\beta}}_2 v_j)^2]/2}{[\sum_{i=1}^I (X_i - \hat{\alpha}_1 - \hat{\beta}_1 u_i)^2 + \sum_{j=1}^J (Y_j - \hat{\alpha}_2 - \hat{\beta}_2 v_j)^2]/(I+J-4)} \\ &= \frac{\frac{IJ}{I+J}\left\{(\hat{\beta}_1 - \hat{\beta}_2)^2 + (\hat{\alpha}_1 - \hat{\alpha}_2)^2\right\}/2}{[\sum_{i=1}^I (X_i - \hat{\alpha}_1 - \hat{\beta}_1 u_i)^2 + \sum_{j=1}^J (Y_j - \hat{\alpha}_2 - \hat{\beta}_2 v_j)^2]/(I+J-4)} \end{aligned}$$

and the test becomes "reject H if $F > F_{2,I+J-4,\alpha}$.

3. In class in the context of Example 6.3.14 we developed a UMP invariant test under normal hypotheses: "reject H if $T \equiv \sqrt{n}\overline{X}/S > t_{n-1,\alpha}(\delta_0)$ where $\delta_0\sqrt{n}\Phi^{-1}(1 - p_0)$.
(a) Study the limiting power of this test assuming that the Y's (and hence also the X's) have $E(Y^2) < \infty$ and that the Y's are i.i.d. according to the location-scale family $F_{\mu,\sigma}(x) = F_0((x-\mu)/\sigma)$. (You will need to decide on how to specify "local alternatives".)
(b) Now consider the alternative test based on the empirical d.f. of the Y's: "reject $H: p \geq p_0$ (in favor of $K:\ p < p_0$) if $n\mathbb{F}_n(y_0) \leq c_{n,\alpha}$ where $c_{n,\alpha}$ is the

largest integer satisfying $P(\text{Bin}(n, p_0) \le c_{n,\alpha}) < \alpha$. Study the limiting power of this test assuming local alternatives of the form $p_n = p_0 - c/\sqrt{n}$ with $c > 0$.
(c) Compare the asymptotic power of the tests in (a) and (b) assuming that F_0 is normal. [Hint: it might be helpful to read example 6.4.2 on page 33 of section 6.4.]

Solution: (a) As discussed in class on 4/7, the first step is to understand the limiting behavior of the critical points under normal theory. We know that under X_i's i.i.d. $N(\theta_0, \sigma_0^2)$ it follows that

$$\begin{pmatrix} \sqrt{n}(\overline{X} - \theta_0)/\sigma_0 \\ \sqrt{n}(S_n^2/\sigma_0^2 - 1) \end{pmatrix} \to_d N_2\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right).$$

Since under normality we have, with $c_0 \equiv \theta_0/\sigma_0 = \Phi^{-1}(1 - p_0)$,

$$\begin{aligned} \alpha &= P_{\theta_0,\sigma_0}\left(\frac{\sqrt{n}\overline{X}}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right) \\ &= P_{\theta_0,\sigma_0}\left(\frac{\sqrt{n}(\overline{X} - \theta_0)}{S_n} + \frac{\sqrt{n}\theta_0}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right) \qquad (1) \\ &= P_{\theta_0,\sigma}\left(\frac{\sqrt{n}(\overline{X} - \theta_0)}{S_n} + \frac{\theta_0}{\sigma_0}\sqrt{n}\left(\frac{\sigma_0}{S_n} - 1\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}c_0\right). \qquad (2) \end{aligned}$$

Now by the delta-method,

$$\begin{pmatrix} \sqrt{n}(\overline{X} - \theta_0)/\sigma_0 \\ \sqrt{n}(\sigma_0/S_n - 1) \end{pmatrix} \to_d \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N_2\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\right).$$

and hence

$$\frac{\sqrt{n}(\overline{X} - \theta_0)}{S_n} + \frac{\theta_0}{\sigma_0}\sqrt{n}\left(\frac{\sigma_0}{S_n} - 1\right) \to_d Z_1 + c_0 Z_2 \sim N(0, 1 + (1/2)c_0^2).$$

Letting $\gamma \equiv \lim_{n\to\infty}\{t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}c_0\}$, and taking limits across the identity in (2) yields

$$\alpha = P(Z_1 + c_0^2 Z_2 > \gamma) = P(\sqrt{1 + (1/2)c_0^2}Z > \gamma)$$

where $Z \sim N(0, 1)$, and this forces $\gamma = \sqrt{1 + (1/2)c_0^2}z_\alpha$ with $z_\alpha = \Phi^{-1}(1 - \alpha)$.

Now we are ready to treat the behavior of the size and power of this test when $F_{\mu,\sigma}(x) = F_0((x - \mu)/\sigma)$ and F_0 is a distribution with mean 0, variance 1, and finite fourth moment. This implies that the X_i's are i.i.d. with mean θ and variance σ^2. From problem 3, problem set #3, Stat 581, we know that

$$\sqrt{n}\begin{pmatrix} \overline{X}_n - \theta \\ S_n^2 - \sigma^2 \end{pmatrix} \to_d \underline{Z} \sim N_2(0, \Sigma)$$

where

$$\begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}.$$

Furthermore, from the solution of part (b) of the same problem we know that with $g(u,v) = u/\sqrt{v}$

$$\begin{aligned} \sqrt{n}(\overline{X}_n/S_n - \theta/\sigma) &= \sqrt{n}(g(\overline{X}_n, S_n^2) - g(\theta,\sigma^2)) \\ &\to_d \nabla g \cdot \underline{Z} \sim N(0, \nabla g^T \Sigma \nabla g) \equiv N(0, V^2) \end{aligned}$$

and it is easy to calculate that

$$\begin{aligned} V^2 &= \nabla g^T \Sigma \nabla g \\ &= \frac{1}{\sigma^4}\left\{\sigma^4 - \theta\mu_3 + \frac{1}{4}c^2(\mu_4 - \sigma^4)\right\} \\ &= 1 - c\gamma_1 + \frac{1}{4}c^2(2+\gamma_2) \equiv V^2(c,\gamma_1,\gamma_2) \equiv V^2 \end{aligned}$$

where $c \equiv \theta/\sigma$, $\gamma_1 \equiv \mu_3/\sigma^3$, and $\gamma_2 \equiv \mu_4/\sigma^4 - 3$. Note that when the X_i's are normal (so $\gamma_1 = \gamma_2 = 0$), this reduces to $1 + c^2/2$. Thus the size of our normal theory test when the Y_i's are really i.i.d. F_{μ_0,σ_0} (with corresponding parameters θ_0 and σ_0 for the X_i's) is

$$\begin{aligned} & P_{\theta_0,\sigma_0}\left(\frac{\sqrt{n}\overline{X}}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right) \\ &= P_{\theta_0,\sigma_0}\left(\sqrt{n}\left(\frac{\overline{X}}{S_n} - \frac{\theta_0}{\sigma_0}\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}c_0\right) \\ &\to P(V_0 Z > \sqrt{1+2^{-1}c_0^2}z_\alpha) = P(Z > V_0^{-1}\sqrt{1+2^{-1}c_0^2}z_\alpha) \\ &= 1 - \Phi(V_0^{-1}\sqrt{1+2^{-1}c_0^2}z_\alpha) \end{aligned}$$

where $V_0^2 \equiv V^2(c_0, \gamma_1(F_0), \gamma_2(F_0))$. When $F_0 = \Phi$ then $\gamma_1(\Phi) = \gamma_2(\Phi) = 0$, $V_0\sqrt{1+2^{-1}c_0^2} = 1$, and the asymptotic size of the test is α (as it must be). However for distributions F_0 with $\gamma_1(F_0) \neq 0$ or $\gamma_2(F_0) \neq 0$, the asymptotic size of this test is generally not α.

How does the power of the normal theory test behave for local alternatives of the form $p_n = p_0 - d/\sqrt{n}$ for $c > 0$? In this case $c_n \equiv \theta_n/\sigma_n = \Phi^{-1}(1-p_n)$ where

$\theta_n \to \theta_0$ and $\sigma_n \to \sigma_0$, and we have

$$\begin{aligned}
\sqrt{n}(c_n - c_0) &= \sqrt{n}(\Phi^{-1}(1-p_n) - \Phi^{-1}(1-p_0)) \\
&= \frac{\Phi^{-1}(1-p_n) - \Phi^{-1}(1-p_0)}{p_n - p_0} \cdot \sqrt{n}(p_n - p_0) \\
&= \frac{d}{du}\Phi^{-1}(1-u)\Big|_{u=p_0} \cdot (-d) \\
&= \frac{-1}{\phi(\Phi^{-1}(1-p_0))}(-d) = \frac{d}{\phi(\Phi^{-1}(1-p_0))}.
\end{aligned}$$

Thus the power of this normal theory test under local alternatives of this form is

$$\begin{aligned}
\beta_{\text{Norm}}(\theta_n, \sigma_n) &= P_{\theta_n,\sigma_n}\left(\frac{\sqrt{n}\overline{X}}{S_n} > t_{n-1,\alpha}(\sqrt{n}c_0)\right) \\
&= P_{\theta_n,\sigma_n}\left(\sqrt{n}\left(\frac{\overline{X}}{S_n} - \frac{\theta_n}{\sigma_n}\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}\frac{\theta_n}{\sigma_n}\right) \\
&= P_{\theta_n,\sigma_n}\left(\sqrt{n}\left(\frac{\overline{X}}{S_n} - \frac{\theta_n}{\sigma_n}\right) > t_{n-1,\alpha}(\sqrt{n}c_0) - \sqrt{n}c_0 - \sqrt{n}(c_n - c_0)\right) \\
&\to P(V_0 Z > (1 + 2^{-1}c_0^2)^{1/2} z_\alpha - d/\phi(\Phi^{-1}(1-p_0))).
\end{aligned}$$

When $F_0 = \Phi$, $V_0^2 = (1 + 2^{-1}c_0^2)$, and the limiting power becomes, under normality,

$$P\left(Z > z_\alpha - \frac{d}{(1 + 2^{-1}c_0^2)^{1/2}\phi(\Phi^{-1}(1-p_0))}\right).$$

In the terminology used in section 6.4, the normal theory test has *efficacy*

$$\epsilon_{Norm} = \frac{1}{(1 + 2^{-1}c_0^2)^{1/2}\phi(\Phi^{-1}(1-p_0))} = \frac{1}{(1 + 2^{-1}\Phi^{-1}(1-p_0)^2)^{1/2}\phi(\Phi^{-1}(1-p_0))}$$

under normality.

(b) If we use the alternative test "reject $H: \ p \ge p_0$ when $n\mathbb{F}_n(y_0) \le c_{n,\alpha}$ where $P(\text{Bin}(n, p_0) \le c_{n,\alpha}) < \alpha$ (and $c_{n,\alpha}$ is the largest such integer), then since

$$\frac{n\mathbb{F}_n(y_0) - np_0}{\sqrt{np_0(1-p_0)}} \to_d Z \sim N(0,1)$$

we deduce that $(c_{n,\alpha} - np_0)/\sqrt{np_0(1-p_0)} \to z_\alpha \equiv \Phi^{-1}(\alpha)$. Thus the power of the binomial test under alternatives of the form $p_n = p_0 - c/\sqrt{n}$ is

$$\begin{aligned}
\beta_{\text{Bin}}(p_n) &= P_{p_n}(n\mathbb{F}_n(y_0) \le c_{n,\alpha}) \\
&= P_{p_n}\left(\frac{n\mathbb{F}_n(y_0) - np_n}{\sqrt{np_n(1-p_n)}} \le \frac{c_{n,\alpha} - np_n}{\sqrt{np_n(1-p_n)}}\right) \\
&= P_{p_n}\left(\frac{n\mathbb{F}_n(y_0) - np_n}{\sqrt{np_n(1-p_n)}} \le \frac{c_{n,\alpha} - np_0}{\sqrt{np_0(1-p_0)}}\frac{\sqrt{np_0(1-p_0)}}{\sqrt{np_n(1-p_n)}} + \frac{\sqrt{n}c}{\sqrt{np_n(1-p_n)}}\right) \\
&\to P\left(Z \le z_\alpha + \frac{c}{\sqrt{p_0(1-p_0)}}\right).
\end{aligned}$$

Thus the *efficacy* of the binomial test is

$$\epsilon_{Bin} = \frac{1}{\sqrt{p_0(1-p_0)}}.$$

(c) As discussed in section 6.4, one convenient way to summarize the comparison between the two tests in (a) and (b) is via their *Pitman efficiency*, which as noted on page 35 is the ratio of the squares of the efficacies. Here we get

$$e_{Bin,Norm} = \left\{\frac{\frac{1}{\sqrt{p_0(1-p_0)}}}{\frac{1}{(1+2^{-1}\Phi^{-1}(1-p_0)^2)^{1/2}\phi(\Phi^{-1}(1-p_0))}}\right\}^2 = \frac{(1+\Phi^{-1}(1-p_0)^2/2)\phi^2(\Phi^{-1}(1-p_0))}{p_0(1-p_0)}.$$

Here is a plot of this relative efficiency as a function of p_0. It is fairly flat, with values in the range of .63 to about .65 over the interval $.15 \le p_0 \le 1 - p_0$, but declines rapidly to 0 for $p_0 \le .15$ or $p_0 \ge .85$. Note that the value at $p_0 = .5$ is $2/\pi$, which is the Pitman efficiency of the sign test relative to the $t-$test at normality. Of course the normal theory test does not have asymptotically correct size when the assumption of normality is violated, while the binomial test maintains the correct size asymptotically regardless of the underlying distribution of the Y's (and hence of the X's).

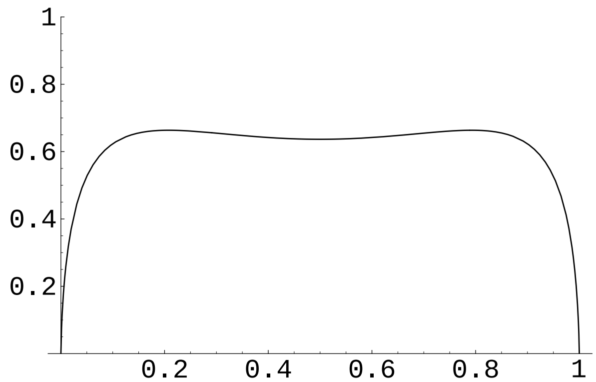

Figure 1: Pitman ARE plot as a function of p_0