Statistics 583, Problem Set 1 Solutions

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1. Let X and Y be independent exponential random variables with parameters λ and μ respectively: thus $P(X > x) = \exp(-\lambda x)$ and $P(Y > y) = \exp(-\mu y)$ for $x, y \ge 0$. Let $\theta \equiv \lambda/\mu$.
(a) Show that the problem of testing $H_0 : \theta \le 1$ versus $H_1 : \theta > 1$ is invariant under the group G of transformations $g_c(x, y) = (cx, cy)$, $c > 0$, and find a UMP invariant test of size α.
(b) Show that the problem of testing $H_0' : \theta = 1$ versus $H_1' : \theta \ne 1$ is invariant *in addition* under the transformation $g(x, y) = (y, x)$, and find a UMP invariant test of size α.
(c) Find UMP invariant tests of the hypotheses in (a) and (b) when $X_1, \ldots, X_m$ are i.i.d. Exponential(λ) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential(μ).

Solution: (a) Now X, Y have joint density

$$p_{\lambda,\mu}(x, y) = \lambda e^{-\lambda x} \mu e^{-\mu y} 1_{[0,\infty)}(x) 1_{[0,\infty)}(y)$$

so that if $c > 0$, $g_c(X, Y) = (cX, cY) \sim p_{\lambda/c,\mu/c}(x, y)$ and hence $\overline{g}(\lambda, \mu) = (\lambda/c, \mu/c)$. Note that $\delta(\lambda, \mu) = \lambda/\mu = (\lambda/c)/(\mu/c) = \delta(\overline{g}(\lambda, \mu))$, so that the hypotheses $H : \delta \le 1$ and $K : \delta > 1$ are invariant. $T(X, Y) = X/Y$ and $\delta(\lambda, \mu) = \lambda/\mu$ are maximmal invariants on the sample space and parameter space respectively, and since $2\mu Y \sim \chi_2^2$ and $2\lambda X \sim \chi_2^2$ are independent

$$T(X, Y) = \frac{2\lambda X}{2\mu Y} \frac{\mu}{\lambda} \sim \delta^{-1} F_{2,2}.$$

Since this family of distributions has monotone (decreasing) likelihood ratio, it follows that the UMP $G-$invariant test of H versus K rejects H when $T < F_{2,2,\alpha}$ (where $P(F_{2,2} \le F_{2,2,\alpha}) = \alpha$. or when $T^{-1} = Y/X > F_{2,2,1-\alpha} = (1 - \alpha)/\alpha$.
(b) If $g(x, y) = (y, x)$ is also considered, then the transformation induced on the maximal invariant of A is given by

$$g(T) = T(g(X, Y)) = T(Y, X) = \frac{Y}{X} = \frac{1}{T(X, Y)}.$$

Claim: $T \vee T^{-1}$ is the maximal invariant with respect to this new group.
Proof. $T \vee T^{-1}$ is invariant since $T^{-1} \vee T = T \vee T^{-1}$; and $T \vee T^{-1}$ is maximal since $T \vee T^{-1} = T^* \vee T^{*-1}$ implies that either $T = T^*$ or $T = T^{*-1}$.

A corresponding maximal invariant on the parameter space is $\delta \vee \delta^{-1} = \frac{\lambda}{\mu} \vee \frac{\mu}{\lambda} \equiv \nu$, and the hypotheses $H : \delta = 1$ and $K : \delta \neq 1$ are clearly invariant. When expressed in terms of δ the hypotheses become $H : \delta = 1$ versus $K : \delta > 1$. It remains to show that the maximal invariant has monotone likelihood ratio (MLR).

By direct calculation, for $t > 1$,

$$1 - F_\eta(t) \;=\; P_{\lambda,\mu}(T \vee T^{-1} > t) = \frac{2 + \eta t}{1 + \eta t + t^2} \tag{1}$$

with $\eta = \delta + 1/\delta$. Hence $T \vee T^{-1}$ has density

$$f_\eta(t) = \frac{\eta(t^2 + 1) + 4t}{(t^2 + \eta t + 1)^2} 1_{[1,\infty)}(t).$$

so that for $\eta_1 < \eta_2$

$$\frac{f_{\eta_2}}{f_{\eta_1}}(t) = \left(\frac{t^2 + \eta_1 t + 1}{t^2 + \eta_2 t + 1} \right)^2 \frac{\eta_2(t^2 + 1) + 4t}{\eta_1(t^2 + 1) + 4t} \equiv g(t)^2 h(t)$$

where

$$g'(t) = \frac{(\eta_2 - \eta_1)(t^2 - 1)}{(t^2 + \eta_2 t + 1)^2} \geq 0 \qquad \text{for} \ \ t \geq 1$$

and

$$h'(t) = \frac{(\eta_2 - \eta_1)(4t^2 - 1)}{(t^2 + \eta_2 t + 1)^2} \geq 0 \qquad \text{for} \ \ t \geq 1.$$

Hence the distribution of $T \vee T^{-1}$ has MLR and the UMP $G-$invariant test rejects H when $M \equiv T \vee T^{-1} > \frac{2-\alpha}{\alpha}$. Then, when $\delta = 1$, $\eta = 2$, and

$$P_{\eta=2}(M > (2 - \alpha)/\alpha)) = \alpha.$$

[Proof of (1): Since $T \sim \delta^{-1} F_{2,2}$ where $P(F_{2,2} \leq x) = x/(1 + x)$,

$$\begin{aligned} P(T \vee T^{-1} > t) &= P(T > t) + P(T < t^{-1}) = P(F_{2,2} > \delta t) + P(F_{2,2} < \delta/t) \\ &= \frac{1}{1 + \delta t} + \frac{\delta/t}{1 + \delta/t} \\ &= \frac{1/\delta}{1/\delta + t} + \frac{\delta}{\delta + t} \\ &= \frac{2 + (\delta + 1/\delta)t}{1 + (\delta + 1/\delta)t + t^2}.] \end{aligned}$$

(c) When $X_1, \ldots, X_m$ are i.i.d. exponential(λ) and $Y_1, \ldots, Y_n$ are i.i.d. exponential(μ) respectively, and $G = \{g : g(\underline{x}, \underline{y}) = (c\underline{x}, c\underline{y}), c > 0\}$, then we reduce by sufficiency first: $(\sum_1^m X_i, \sum_1^n Y_j)$ is sufficient for (λ, μ). Then $T = \sum_1^m X_i / \sum_1^n Y_j$ is a maximal invariant with respect to $G^* = \{g^* : g^*(x, y) = (cx, cy),\ c > 0\}$ acting on the space of the sufficient statistic. Moreover,

$$T = \frac{\mu}{\lambda}\frac{m}{n}\frac{2\lambda \sum_1^m X_i/(2m)}{2\mu \sum_1^n Y_j/(2n)} \sim \frac{1}{\delta n/m} F_{2m,2n}$$

where $\delta \equiv \delta(\lambda, \mu) = \lambda/\mu$ is a maximal invariant with respect to $\overline{G}$ on the parameter space, and the density of $F_{2m,2n}$ is given by

$$f_{F_{2m,2n}}(x) = c_{m,n} \frac{x^{m-1}}{(1 + (m/n)x)^{m+n}} 1_{(0,\infty)}(x).$$

Thus the density of T is given by

$$f_T(x, \delta) = \delta \tilde{c}_{m,n} \frac{(\delta x)^{m-1}}{(1 + \delta x)^{m+n}} 1_{(0,\infty)}(x)$$

which has monotone (decreasing) likelihood ratio in $M(x) = x$. Thus the UMP-G^* invariant test rejects H_0 when $((n/m)T)^{-1} = (\sum_1^n Y_j/2n)/(\sum_1^m X_i/2m) > F_{2n,2m,\alpha}$ where $P(F_{2n,2m} > F_{2n,2m,\alpha}) = \alpha$.

For testing $H_0' : \theta = 1$ versus $H_1' : \theta \neq 1$, we find, much as in (b), that $\tilde{T} \equiv T \vee T^{-1}$ is a maximal invariant with respect to the induced group $G^* = G_2^* \oplus G_1^*$ on the space of the sufficient statistic. By a calculation similar to that of part (b),

$$\begin{aligned}
1 - F(t) &= P(T \vee T^{-1} > t) = P(T > t) + P(T < 1/t) \\
&= P(\delta^{-1}(m/n)F_{2m,2n} > t) + P(\delta^{-1}(m/n)F_{2m,2n} < 1/t) \\
&= P((m/n)F_{2m,2n} > \delta t) + P((m/n)F_{2m,2n} < \delta/t)
\end{aligned}$$

and hence

$$\begin{aligned}
f_{T\vee T^{-1}}(t) &= \delta \tilde{c}_{m,n} \frac{(\delta t)^{m-1}}{(1 + \delta t)^{m+n}} + \frac{\delta}{t^2}\tilde{c}_{m,n}\frac{(\delta/t)^{m-1}}{(1 + \delta/t)^{m+n}} \\
&= \tilde{c}_{m,n}\frac{t^{m-1}(\delta + t)^m (1 + t/\delta)^n + t^{n-1}(1 + \delta t)^m (1/\delta + t)^n}{(t^2 + (\delta + 1/\delta)t + 1)^{m+n}}.
\end{aligned}$$

It is not yet clear to me that this density depends only on $\eta = \delta + 1/\delta$, so I may be making a mistake somewhere.

2. (Continuation of problem 4(f), Stat 582 final exam.) Suppose that $X \sim$ Binomial(m, p_1) and $Y \sim$ Binomial(n, p_2) are independent. In problem 4 of the 582 final exam we

derived the UMP unbiased (conditional) test of level α for testing $H : p_2 \leq p_1$ versus $K : p_2 > p_1$. It involves rejecting H if $Y > c(t)$ relative to the conditional (hypergeometric) distribution of Y conditional on $T = X+Y = t$, or equivalently if

$$\frac{Y/t - (n/N)}{\sigma_N} > c'(t)$$

where $c'(T) \to_p z_\alpha \equiv \Phi^{-1}(1-\alpha)$ since

$$\frac{Y/t - (n/N)}{\sigma_N} \to_d Z \sim N(0,1)$$

conditionally on T if $0 < \liminf(n/N) \leq \limsup(n/N) < 1$. Here $\sigma_N^2 = (1-(t-1)/(N-1))(n/N)(1-n/N)/t$.

(a) Show that

$$\frac{Y/t - (n/N)}{\sigma_N} = \frac{\sqrt{\frac{mn}{N}}(\frac{Y}{n} - \frac{X}{m})}{\sqrt{\frac{t}{N}(1 - \frac{t-1}{N-1})}}.$$

(b) Use the result of (a) to show that

$$\frac{Y/T - (n/N)}{\sigma_N} \to_d Z \sim N(0,1)$$

unconditionally under $p_1 = p_2 \equiv p$ (even if $a = \liminf(n/N) < \limsup(n/N) = b$ with possibly $a = 0$ or $b = 1$). [Hint: prove it first under the assumption that $n/N \to \lambda \in [0,1]$, and then show that the limit is the same even if n/N does not converge by considering subsequences.]

(c) What is the (unconditional) limiting behavior of the test statistic $(Y/t - (n/N))/\sigma_N$ under local alternatives of the form $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$ assuming that $n/N \to \lambda \in (0,1)$?

(d) What does the result of (c) imply about the limiting power of the test under these alternatives?

Solution: (a) Note that

$$\begin{aligned} \frac{Y}{T} - \frac{n}{N} &= \frac{1}{T}\left(Y - \frac{n}{N}T\right) = \frac{1}{T}\left(Y - \frac{n}{N}(X+Y)\right) \\ &= \frac{1}{T}\left(\frac{m}{N}Y - \frac{n}{N}X\right) = \frac{mn}{NT}\left(\frac{Y}{n} - \frac{X}{m}\right). \end{aligned}$$

Therefore

$$\begin{aligned}
\frac{Y/t-(n/N)}{\sigma_N} &= \frac{Y/t-n/N}{\sqrt{(1-\frac{n-1}{N-1})\frac{n}{N}\frac{m}{N}\frac{1}{t}}} = \frac{\sqrt{t}(Y/t-n/N)}{\sqrt{(1-\frac{n-1}{N-1})\frac{n}{N}\frac{m}{N}}} \\
&= \frac{\frac{mn}{N}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{T(1-\frac{T-1}{N-1})\frac{mn}{N^2}}} = \frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}}
\end{aligned}$$

as claimed.

(b) Now $(\sqrt{m}(X/m-p), \sqrt{n}(Y/n-p)) \to_d (Z_1, Z_2)$ where $Z_j \sim N(0, p(1-p))$, $j=1,2$ are independent. Thus from (a), under the assumption that $n/N \to \lambda \in [0,1]$,

$$\begin{aligned}
V_N \equiv \frac{Y/T-(n/N)}{\sigma_N} &= \frac{\sqrt{\frac{mn}{N}}\left(\frac{Y}{n}-\frac{X}{m}\right)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}} \\
&= \frac{\sqrt{\frac{m}{N}}\sqrt{n}(Y/n-p) - \sqrt{\frac{n}{N}}\sqrt{m}(X/m-p)}{\sqrt{\frac{T}{N}(1-\frac{T-1}{N-1})}} \\
&\to_d \frac{\sqrt{1-\lambda}Z_2 - \sqrt{\lambda}Z_1}{\sqrt{p(1-p)}} \sim N(0,1)
\end{aligned}$$

where we used $T/N = (m/N)(X/m) + (n/N)(Y/n) \to_p (1-\lambda)p + \lambda p = p$.

If $\lambda_N \equiv n/N \not\to$, then since $\lambda_N \in [0,1]$, for any initial subsequence $\{\lambda_{N'}\}$, there exists a further convergent subsequence $\{\lambda_{N''}\}$; i.e. $\lambda_{N''} \to$ some $\lambda \in [0,1]$. By the same argument as above, for this subsequence $V_{N''} \to_d Z \sim N(0,1)$. Since the limiting distribution is the same for any such initial subsequence $\{V_{N'}\}$, we conclude that the full sequence $\{V_N\}$ satisfies $V_N \to_d Z \sim N(0,1)$ under $p_1 = p_2 = p$. (This argument is completely analogous to the following fact concerning real numbers: a sequence $\{x_n\}$ of real numbers satisfies $x_n \to x$ if and only if each subsequence $\{x_{n'}\}$ contains a further subsequence $\{x_{n''}\}$ such that $x_{n''} \to x$. See Billingsley (1968), *Convergence of Probability Measures*, theorem 2.3, page 16.)

(c) Under local alternatives $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$, we have

$$\begin{aligned}
\sqrt{n}(Y/n-p_1) &= \sqrt{n}(Y/n-p_1-c/\sqrt{N}) + \sqrt{n}c/\sqrt{N} \\
&= \sqrt{n}(Y/n-p_{2,N}) + c\sqrt{\frac{n}{N}} \\
&\to_d Z_2 + c\sqrt{\lambda} \sim N(c\sqrt{\lambda}, p_1(1-p_1))
\end{aligned}$$

under the assumption that $\lambda_N = n/N \to \lambda$. Then it follows that $(\sqrt{m}(X/m - p_1), (\sqrt{n}(Y/n - p_1)) \to_d (Z_1, Z_2 + c\sqrt{\lambda})$ and hence

$$V_N \to_d \frac{\sqrt{1-\lambda}(Z_2 + c\sqrt{\lambda}) - \sqrt{\lambda} Z_1}{\sqrt{p_1(1-p_1)}} \sim N\left(c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}}\ ,\ 1\right).$$

(d) The limiting distribution under local alternatives found in (c) implies that the power of the test based on V_N statisfies

$$\begin{aligned}
\lim_{N\to\infty} \beta((p_1, p_{2,N})) &= \lim_{N\to\infty} P_{(p_1,p_{2,N})}(V_N > z_\alpha) \\
&= P\left(Z + c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}} > z_\alpha\right) \\
&= 1 - \Phi\left(z_\alpha - c\sqrt{\frac{\lambda(1-\lambda)}{p_1(1-p_1)}}\right).
\end{aligned}$$

3. Let $\Theta = \{(\Delta, \nu) : \ \Delta \in \mathbb{R},\ 1 \le \nu \le n,\ \nu \text{ an integer}\}$ and let the distribution of $(X_1, \ldots, X_n)$, given $\theta = (\Delta, \nu)$, be as independent random variables with $X_i \in N(0,1)$ for $i \ne \nu$, and $X_\nu \sim N(\Delta, 1)$. Test the hypothesis $H_0 : \Delta = 0$ against alternatives $H_1 : \Delta > 0$ or $\overline{H}_1 : \Delta \ne 0$.
(a) Show that this problem is invariant under the group of permutations of $(X_1, \ldots, X_n)$ and that the distribution of the maximal invariant $Y \equiv (Y_1, \ldots, Y_n) = (X_{(1)}, \ldots, X_{(n)})$ (the order statistics) has density

$$f_{\underline{Y}}(y_1, \ldots, y_n | \Delta) \ = \ (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\sum_1^n y_i^2 - \frac{1}{2}\Delta^2\right)(n-1)! \sum_{\nu=1}^n \exp(\Delta y_\nu)$$

for $y_1 < y_2 < \cdots < y_n$ and zero elsewhere.
(b) Show that the locally best invariant test of H_0 versus H_1 is to reject H_0 if $\sum_{i=1}^n X_i$ is too large.
(c) Show that the locally best unbiased invariant test of H_0 versus $\overline{H}_1$ is to reject H_0 if $\sum_1^n X_i^2$ is too large.

Solution: (a) If $g(x) = \pi x = (x_{\pi(1)}, \ldots, x_{\pi(n)})$, for $\pi \in \Pi$, then for $X \sim P_{(\Delta,\nu)}$, it follows that $g(X) = (X_{\pi(1)}, \ldots, X_{\pi(n)}) \sim P_{(\Delta, \pi^{-1}(\nu))}$, so $\overline{g}\theta = \overline{g}(\Delta, \nu) = (\Delta, \pi^{-1}(\nu))$. Thus testing H_0 versus H_1 or $\overline{H}_1$ is clearly invariant. The maximal invariant of the group G of permutations is the vector of order statistics $Y = (Y_1, \ldots, Y_n) \equiv (X_{(1)}, \ldots, X_{(n)})$. From problem set 9, problem 4, Stat 582, the density of Y is given by

$$\begin{aligned}
f_Y(y|\Delta) &= \sum_{\pi\in\Pi} f_X(\pi y;\Delta,\nu)\\
&= \sum_{\pi\in\Pi}\prod_{i\neq\nu}(2\pi)^{-1/2}\exp(-y_{\pi(i)}^2/2)(2\pi)^{-1/2}\exp(-(y_{\pi(\nu)}-\Delta)^2/2)\\
&= (2\pi)^{-n/2}\sum_{\pi\in\Pi}\prod_{i\neq\nu}\exp(-y_{\pi(i)}^2/2)\exp(-y_{\pi(\nu)}^2/2-\Delta^2/2)\exp(\Delta y_{\pi(\nu)})\\
&= (2\pi)^{-n/2}\sum_{\pi\in\Pi}\exp\left(-\sum_{i=1}^n y_{\pi(i)}^2/2-\Delta^2/2\right)\exp(\Delta y_{\pi(\nu)})\\
&= (2\pi)^{-n/2}\exp\left(-\frac{1}{2}\sum_{j=1}^n y_j^2-\frac{1}{2}\Delta^2\right)\sum_{\pi\in\Pi}\exp(\Delta y_{\pi(\nu)})\\
&= (2\pi)^{-n/2}\exp\left(-\frac{1}{2}\sum_{j=1}^n y_j^2-\frac{1}{2}\Delta^2\right)(n-1)!\sum_{j=1}^n\exp(\Delta y_j)
\end{aligned}$$

for $y_1 < y_2 < \cdots < y_n$ and zero elsewhere.

(b) The locally best test of H_0 versus H_1 rejects for large values of

$$\begin{aligned}
&\frac{\partial}{\partial\Delta}\log f_Y(Y|\Delta)\Big|_{\Delta=0}\\
&\quad= \frac{\partial}{\partial\Delta}\left\{\text{const.}-\frac{1}{2}\Delta^2+\ \log\left(\sum_{j=1}^n\exp(\Delta Y_j)\right)\right\}\Big|_{\Delta=0}\\
&\quad= \left\{-\Delta+\frac{\sum_{j=1}^n Y_j\exp(\Delta Y_j)}{\sum_{j=1}^n\exp(\Delta Y_j)}\right\}\Big|_{\Delta=0}\\
&\quad= \frac{1}{n}\sum_{j=1}^n Y_j = \frac{1}{n}\sum_{i=1}^n X_i.
\end{aligned}$$

(c) A test ϕ_0 is locally best unbiased of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ if among all tests ϕ satisfying $\beta_\phi(\theta_0) = \alpha$ and $\beta'_\phi(\theta_0) = 0$, the test ϕ_0 maximizes the value of the second derivative of the power function at θ_0; i.e. if

$$\beta''_{\phi_0}(\theta_0) \geq \beta''_\phi(\theta_0).$$

By the generalized Neyman - Pearson lemma, any test ϕ_0 of the form

$$\phi_0(x) = \begin{cases} 1 & \text{if } \frac{\partial^2}{\partial\theta^2}f(x,\theta)\Big|_{\theta=\theta_0} > k_1 f(x,\theta_0)+k_2\frac{\partial}{\partial\theta}f(x,\theta)\Big|_{\theta=\theta_0}\\ \gamma(x) & \text{if } \frac{\partial^2}{\partial\theta^2}f(x,\theta)\Big|_{\theta=\theta_0} = k_1 f(x,\theta_0)+k_2\frac{\partial}{\partial\theta}f(x,\theta)\Big|_{\theta=\theta_0}\\ 0 & \text{if } \frac{\partial^2}{\partial\theta^2}f(x,\theta)\Big|_{\theta=\theta_0} < k_1 f(x,\theta_0)+k_2\frac{\partial}{\partial\theta}f(x,\theta)\Big|_{\theta=\theta_0} \end{cases}$$

where k_1 and k_2 are chosen so that the resulting test satisfies the size and unbiasedness restrictions will be locally best unbiased. Using

$$\frac{\partial}{\partial\theta}\log f(x,\theta) = \frac{\frac{\partial}{\partial\theta}f(x,\theta)}{f(x,\theta)}$$

and noticing that

$$\frac{\frac{\partial^2}{\partial\theta^2}f(x,\theta)}{f(x,\theta_0)}\bigg|_{\theta=\theta_0} = \frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\bigg|_{\theta=\theta_0} + \left(\frac{\partial}{\partial\theta}\log f(x,\theta)\right)^2\bigg|_{\theta=\theta_0}$$

for points x with $f(x,\theta_0) > 0$, it follows that rejecting when

$$\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\bigg|_{\theta=\theta_0} + \left(\frac{\partial}{\partial\theta}\log f(x,\theta)\right)^2\bigg|_{\theta=\theta_0} > k_1 + k_2\frac{\partial}{\partial\theta}\log f(x,\theta)\bigg|_{\theta=\theta_0}$$

yields a locally most powerful unbiased test.

In our present situation we seek a locally most powerful unbiased invariant test, so we need to carry out the above calculations for the density of the maximal invariant, namely the density of the order statistics derived in (a). We already know from (b) that

$$\frac{\partial}{\partial\Delta}\log f_Y(Y|\Delta)\bigg|_{\Delta=0} = \overline{X},$$

and we easily calculate

$$\begin{aligned}
&\frac{\partial^2}{\partial\Delta^2}\log f_Y(Y|\Delta)\bigg|_{\Delta=0} \\
&= -1 + \frac{\sum_1^n Y_j^2\exp(\Delta Y_j)}{\sum_1^n \exp(\Delta Y_j)} - \left(\frac{\sum_1^n Y_j\exp(\Delta Y_j)}{\sum_1^n \exp(\Delta Y_j)}\right)^2\bigg|_{\Delta=0} \\
&= -1 + \frac{1}{n}\sum_1^n Y_j^2 - \overline{Y}^2 = -1 + \frac{1}{n}\sum_1^n X_i^2 - \overline{X}^2.
\end{aligned}$$

Thus the locally best unbiased invariant test rejects if

$$\frac{1}{n}\sum_1^n X_j^2 > k_1 + k_2\overline{X}$$

where k_1, k_2 are chosen to satisfy the size and unbiasedness restrictions. But it is easily seen that the unbiasedness restriction can be written as

$$E_0\{\phi_0(X)\overline{X}\} = 0.$$

But by symmetry considerations (as discussed in Ferguson, pages 239 - 240), this forces $k_2 = 0$, and hence the locally best unbiased invariant test rejects for large values of $\sum_1^n X_i^2$.