Statistics 583, Problem Set 6

Wellner; 5/3/2006

Reading: Chapter 8, sections 8.1- 8.4, pages 1 - 18; Wasserman, Chapters 2-3, pages 13-41.
Due: Wednesday, May 10, 2006
Reminder 1: Fourth (and last) make-up lecture, Monday, May 8, 9:30 AM.
Reminder 2: Midterm exam, Friday, May 5.

1. Show that for linear statistics the jackknife and bootstrap estimates of bias are zero. (A linear statistic corresponds to a functional $T(F)$ of the form $T(F) = \int \psi dF$ for some function ψ for a distribution function F on R, or $T(P) = \int \psi dP$ for a probability distribution P on a general sample space $\mathcal{X}$.)

2. (a) Given n distinct data items, show that the probability that a given data item does not appear in a bootstrap sample is $e_n = (1 - 1/n)^n$
(b) Show that $e_n \to e^{-1} \approx .368$ as $n \to \infty$.
(c) Hence show that the probability that each of B bootstrap samples contains an item i is $(1 - e_n)^B$. Evaluate this quantity for $n = 10, 20, 50, 100$ and $B = 10, 20, 50, 100$.
(d) Let $N_n \equiv \sum_{j=1}^n 1_{[M_j=0]}$ where $\underline{M} \equiv (M_1, \ldots, M_n) \sim$ Mult$_n(n, \underline{1}/n)$. Show that $E(n^{-1}N_n) = e_n$ as computed in (a).

3. Suppose that $T(F) = Var_F(X)$ so that $T_n \equiv T(\mathbb{F}_n) = n^{-1}\sum_{i=1}^n (X_i - \overline{X})^2$. Show that the jackknife estimate of the variance $\sigma_n^2(F) \equiv Var_F(T_n)$ is

$$\widehat{Var} = \frac{n^2}{(n-1)^3}(\widehat{\mu}_4 - \widehat{\mu}_2^2)$$

where $\widehat{\mu}_k \equiv n^{-1}\sum_{i=1}^n (X_i - \overline{X})^k$ for $k = 1, 2, \ldots$. Hence, assuming that $EX^4 < \infty$, the jackknife estimate of variance is consistent for this T:

$$n\widehat{Var} \to_p \mu_4 - \mu_2^2 = \mu_2^2\{2 + \frac{\mu_4}{\mu_2^2} - 3\} = T_2(F)(2 + \gamma_2)\,.$$

4. (a) Wasserman, problem 3.8.9, page 40.
(b) Wasserman, problem 3.8.13, page 41.