Statistics 583, Problem Set 5

Wellner; 4/26/2006

Reading: Chapter 7, sections 7.1- 7.4; Wasserman, Chapters 2-3, pages 13-41.
Due: Wednesday, May 3, 2006
Reminder 1: Third make-up lecture, Monday, May 1, 9:30 AM.
Reminder 2: Midterm exam, Friday, May 5.

1. Suppose that we observe $X_1, \ldots, X_n$ i.i.d. P on $\mathbb{R}^+ = [0, \infty)$ and assume that $P \in \mathcal{P}_0 \equiv \{P_\theta : (dP_\theta/d\lambda) = p_\theta, \ \theta \in \Theta\}$ where $\theta = (\alpha, \beta) \in (0, \infty)^2$ and $p_\theta = p_{\alpha,\beta}$ is the Weibull density given by $p_\theta(x) = (\beta/\alpha)(x/\alpha)^{\beta-1} \exp(-(x/\alpha)^\beta) 1_{(0,\infty)}(x)$. From Lehmann and Romano, TPE, Example 6.6.1 (page 468) and problems 6.6.1 - 6.6.3 (page 509), we know that the maximum likelihood estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ exists and is unique if $0 < X_{(1)} < X_{(n)}$.
(a) If, in fact, $P \notin \mathcal{P}_0$, to what function of P do you expect $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ converges (in probability)? [Hint: use the development in example 6.6.1 of Lehmann and Romano to show that the solution of the population version of the score equations (with sampling from $P \notin \mathcal{P}$) leads to $\alpha(P) = \{E_P(X^\beta)\}^{1/\beta}$ where β is the solution of
$$\frac{E_P(X^\beta \log X)}{E_P X^\beta} - \frac{1}{\beta} = E_P(\log X),$$
assuming that $E_P(X^\beta |\log X|) < \infty$.]
(b) Show heuristically that $\theta(P) = (\alpha(P), \beta(P))$ minimizes $K(P, P_\theta)$ over Θ.
(c) Show in particular that if P has density $p(x) = 2\phi(x) 1_{(0,\infty)}(x)$ where ϕ is the standard normal density (so p is the "half-normal" density), then $(\alpha(P), \beta(P)) = (0.855417\ldots, 1.26276\ldots)$. Plot $p(x)$ and the "best-fitting" member of the Weibull family $p_{(\alpha(P),\beta(P))}(x)$ as functions of x.

2. Suppose that $\hat{\theta}_n$ is the MLE for the Weibull family as in problem 1 above, and that $P \notin \mathcal{P}_0$. Heuristically we expect that
$$\sqrt{n}(\hat{\theta}_n - \theta(P)) \rightarrow_d N_2(0, \Sigma(P)) \qquad (1)$$
for some covariance matrix $\Sigma = \Sigma(P)$ as $n \rightarrow \infty$.
(a) What is the form of Σ that you expect in (1)?
(b) What methods could be used to make these heuristics precise?

3. (a) Exercise 2.7.5, Wasserman, page 24.
(b) What can you say about $T(F) = \int x dF(x)$ using this same line of reasoning when the hypothesis $|X| \leq M < \infty$ in (a) fails?

4. Exercise 3.8.1, Wasserman, page 39. [Hint: see Wasserman, page 29.] Under what additional hypotheses can we establish $\sqrt{n}(T(\mathbb{F}_n) - T(F)) \to N(0, E_F \psi_F^2(X))$? (Here my ψ_F equals Wasserman's L_F.)