Statistics 583, Problem Set 3

Wellner; 4/14/2006

Reading: Chapter 7, sections 7.1- 7.4 (to be handed out on Monday, 4/17); Wasserman, Chapters 1-2, pages 1-24.
Due: Wednesday, April 19, 2006

1. Suppose that an urn contains N balls with the numbers $z_i = -\log(1-i/(N+1))$, $i = 1, \ldots, N$ and we sample $n < N$ balls from this urn. Let $\overline{Y}_n = n^{-1} \sum_1^n Y_i$ denote the sample mean of the sampled valls.
(a) Calculate the mean $\mu_N = E(\overline{Y}_n)$ and variance $\sigma_N^2 = Var(\overline{Y}_n)$ of $\overline{Y}_n$. Find the limits of $\overline{z}_N$ and σ_z^2 as $N \to \infty$.
(b) Use the Wald-Wolfowitz-Noether-Hajek finite-sampling CLT to prove that $(\overline{Y}_n - \mu_N)/\sigma_N \to_d N(0,1)$.
(c) What classical two-sample rank statistic is $\overline{Y}_n$ equivalent to under the null hypothesis (of all $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ equal in distribution with a common continuous distribution function F)?

2. (a) What is the locally best rank test of $F = G$ against $F = (e^{\theta G} - 1)/(e^{\theta} - 1)$, $\theta > 0$?
(b) What is the locally best rank test of $F = G$ against $F = G/(e^{\theta}(1-G)+G)$?
(c) What can you say about the power of these tests?

3. Suppose that $X_1, \ldots, X_n$ are independent Exponential(1) random variables. Let $Y_i \equiv X_{(i)}$, for $i = 1, \ldots, n$, denote the *order statistics* corresponding to $X_1, \ldots, X_n$.
(a) Show that the vector $(Y_1, \ldots, Y_n)$ has the same joint distribution as $(W_1, \ldots, W_n)$ where $W_i \equiv \sum_{j=1}^{i} Z_j/(n-j+1)$ and $Z_1, \ldots, Z_n$ are i.i.d. Exponential(1).
(b) Use the result of (a) to compute $E(Y_i)$, $Var(Y_i)$, and $Cov(Y_i, Y_j)$ for any fixed i, j.