Statistics 583, Problem Set 2

Wellner; 4/5/2006

Reading: Chapter 6, sections 6.3 and 6.4; Lehmann and Romano, TSH, Chapters 6 and 7. See also Ferguson, MS, Chapter 5, sections 5.6 and 5.7;
Due: Friday, April 14, 2006

1. Let X_{ij} for $j = 1, \ldots, n_i$ and $i = 1, \ldots, s$ be independent, normally distributed random variables with common variance σ^2, and suppose that $EX_{ij} = \mu_i$. Consider testing $H: \ \mu_1 = \mu_2 = \cdots = \mu_s$ versus $K: \ \mu_i \neq \mu_j$ for some $i \neq j$. (a) Put this in the form of the general linear model and find the UMP invariant test of H versus K.
(b) What is the distribution of the test statistic under the general hypothesis? [See Lehmann and Romano, section 7.3, pages 285 - 286, and Ferguson, example 5.9.1, page 265.]
(c) Suppose that $s = 4$ and $n_i = 6$ for $i = 1, \ldots, 4$. Suppose further that that observed data yield $\overline{X}_{1\cdot} = 0$, $\overline{X}_{2\cdot} = 4$, $\overline{X}_{3\cdot} = 5$, $\overline{X}_{4\cdot} = 7$ and $\hat{\sigma}^2 = 10$. Compute the F-statistic derived in (a) and the $p-$value for this data for testing H versus K.

2. Let X_i, $i = 1, \ldots, I$ and $Y_j = 1, \ldots, J$ satisfy the general linear hypothesis with $EX_i = \alpha_1 + \beta_1 u_i$ and $EY_j = \alpha_2 + \beta_2 v_j$, where the u_i and v_j are known and $\sum u_i = 0 = \sum v_j$, $\sum u_i^2 = I$ and $\sum v_j^2 = J$.
(a) Find the UMP invariant test of $H_0: \beta_1 = \beta_2$ versus $K_0: \beta_1 \neq \beta_2$.
(b) Find the UMP invariant test of H_0': $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

3. In class in the context of Example 6.3.14 we developed a UMP invariant test under normal hypotheses: "reject H if $T \equiv \sqrt{n}\overline{X}/S > t_{n-1,\alpha}(\delta_0)$ where $\delta_0 \sqrt{n}\Phi^{-1}(1 - p_0)$.
(a) Study the limiting power of this test assuming that the Y's (and hence also the X's) have $E(Y^2) < \infty$ and that the Y's are i.i.d. according to the location-scale family $F_{\mu,\sigma}(x) = F_0((x-\mu)/\sigma)$. (You will need to decide on how to specify "local alternatives".)
(b) Now consider the alternative test based on the empirical d.f. of the Y's: "reject $H: p \geq p_0$ (in favor of $K: \ p < p_0$) if $n\mathbb{F}_n(y_0) \leq c_{n,\alpha}$ where $c_{n,\alpha}$ is the largest integer satisfying $P(\text{Bin}(n, p_0) \leq c_{n,\alpha}) < \alpha$. Study the limiting power of this test assuming local alternatives of the form $p_n = p_0 - c/\sqrt{n}$ with $c > 0$.
(c) Compare the asymptotic power of the tests in (a) and (b) assuming that F_0 is normal. [Hint: it might be helpful to read example 6.4.2 on page 33 of section 6.4.]