Statistics 583, Problem Set 1

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Reading: Chapter 6, sections 6.3 and 6.4; Lehmann and Romano, TSH, Chapters 6 and 7. See also Ferguson, MS, Chapter 5, sections 5.6 and 5.7;
Due: Wednesday, April 5, 2006

1. Let X and Y be independent exponential random variables with parameters λ and μ respectively: thus $P(X > x) = \exp(-\lambda x)$ and $P(Y > y) = \exp(-\mu y)$ for $x, y \geq 0$. Let $\theta \equiv \lambda/\mu$.
(a) Show that the problem of testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$ is invariant under the group G of transformations $g_c(x, y) = (cx, cy)$, $c > 0$, and find a UMP invariant test of size α.
(b) Show that the problem of testing $H_0' : \theta = 1$ versus $H_1' : \theta \neq 1$ is invariant *in addition* under the transformation $g(x, y) = (y, x)$, and find a UMP invariant test of size α.
(c) Find UMP invariant tests of the hypotheses in (a) and (b) when $X_1, \ldots, X_m$ are i.i.d. Exponential(λ) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential(μ).

2. (Continuation of problem 4(f), Stat 582 final exam.) Suppose that $X \sim$ Binomial(m, p_1) and $Y \sim$ Binomial(n, p_2) are independent. In problem 4 of the 582 final exam we derived the UMP unbiased (conditional) test of level α for testing $H : p_2 \leq p_1$ versus $K : p_2 > p_1$. It involves rejecting H if $Y > c(t)$ relative to the conditional (hypergeometric) distribution of Y conditional on $T = X + Y = t$, or equivalently if
$$\frac{Y/t - (n/N)}{\sigma_N} > c'(t)$$
where $c'(T) \to_p z_\alpha \equiv \Phi^{-1}(1 - \alpha)$ since
$$\frac{Y/t - (n/N)}{\sigma_N} \to_d Z \sim N(0, 1)$$
conditionally on T if $0 < \liminf(n/N) \leq \limsup(n/N) < 1$. Here $\sigma_N^2 = (1 - (t - 1)/(N - 1))(n/N)(1 - n/N)/t$.
(a) Show that
$$\frac{Y/t - (n/N)}{\sigma_N} = \frac{\sqrt{\frac{mn}{N}}(\frac{Y}{n} - \frac{X}{m})}{\sqrt{\frac{t}{N}(1 - \frac{t-1}{N-1})}}.$$

(b) Use the result of (a) to show that

$$\frac{Y/T - (n/N)}{\sigma_N} \to_d Z \sim N(0,1)$$

unconditionally under $p_1 = p_2 \equiv p$ (even if $a = \liminf(n/N) < \limsup(n/N) = b$ with possibly $a = 0$ or $b = 1$). [Hint: prove it first under the assumption that $n/N \to \lambda \in [0,1]$, and then show that the limit is the same even if n/N does not converge by considering subsequences.

(c) What is the (unconditional) limiting behavior of the test statistic $(Y/t - (n/N))/\sigma_N$ under local alternatives of the form $p_2 = p_{2,N} = p_1 + c/\sqrt{N}$ assuming that $n/N \to \lambda \in (0,1)$?

(d) What does the result of (c) imply about the limiting power of the test under these alternatives?

3. Let $\Theta = \{(\Delta, \nu) : \Delta \in \mathbb{R},\ 1 \le \nu \le n,\ \nu \text{ an integer}\}$ and let the distribution of $(X_1, \ldots, X_n)$, given $\theta = (\Delta, \nu)$, be as independent random variables with $X_i \in N(0,1)$ for $i \neq \nu$, and $X_\nu \sim N(\Delta, 1)$. Test the hypothesis $H_0 : \Delta = 0$ against alternatives $H_1 : \Delta > 0$ or $\overline{H}_1 : \Delta \neq 0$.

(a) Show that this problem is invariant under the group of permutations of $(X_1, \ldots, X_n)$ and that the distribution of the maximal invariant $Y \equiv (Y_1, \ldots, Y_n) = (X_{(1)}, \ldots, X_{(n)})$ (the order statistics) has density

$$f_{\underline{Y}}(y_1, \ldots, y_n | \Delta) \;=\; (2\pi)^{-n/2} \exp\left(-\frac{1}{2} \sum_1^n y_i^2 - \frac{1}{2}\Delta^2 \right) (n-1)! \sum_{\nu=1}^n \exp(\Delta y_\nu)$$

for $y_1 < y_2 < \cdots < y_n$ and zero elsewhere.

(b) Show that the locally best invariant test of H_0 versus H_1 is to reject H_0 if $\sum_{i=1}^n X_i$ is too large.

(c) Show that the locally best unbiased invariant test of H_0 versus $\overline{H}_1$ is to reject H_0 if $\sum_1^n X_i^2$ is too large.