

Statistics 583, Problem Set 7 Solutions

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1. Let $U_{m,n} \equiv T(\mathbb{F}_m, \mathbb{G}_n)$ where $T(F, G) = \int FdG = P(X \leq Y)$ is the Mann-Whitney functional and \mathbb{F}_m and \mathbb{G}_n are the empirical df's of X_1, \dots, X_m i.i.d. with df F , Y_1, \dots, Y_n i.i.d. with df G .

A. Show that

$$mnU_{m,n} + n(n+1)/2 = W_{m,n} \equiv \sum_{j=1}^n Q_j = \sum_{j=1}^n R_{m+j}.$$

B. Show that $EU_{m,n} = P(X \leq Y) = \int FdG$ and that

$$\begin{aligned} \text{Var}(\sqrt{mn}U_{m,n}) &= (n-1) \int (1-G)^2 dF + (m-1) \int F^2 dG \\ &\quad - (N-1) \left(\int FdG \right)^2 + \int FdG \\ &= (n-1) \text{Var}[1-G(X)] + (m-1) \text{Var}[F(Y)] \\ &\quad + \int FdG(1 - \int FdG). \end{aligned}$$

C. When $F = G$ use the results of A and B to compute $E_{(F,F)}W_{m,n}$ and $\text{Var}_{(F,F)}(W_{m,n})$. (This should agree with calculations for the Wilcoxon rank sum form of the statistic under the null hypothesis via finite sampling calculations.)

Solution: A. Using empirical distribution function notation, $N\mathbb{H}_N = m\mathbb{F}_m + n\mathbb{G}_n$, so

$$\begin{aligned} mnU_{m,n} &= \int m\mathbb{F}_m dn\mathbb{G}_n = \int N\mathbb{H}_N dn\mathbb{G}_n - \int n\mathbb{G}_n dn\mathbb{G}_n \\ &= \sum_{j=1}^n N\mathbb{H}_N(Y_j) - \sum_{j=1}^n n\mathbb{G}_n(Y_j) \\ &= \sum_{j=1}^n R_{m+j} - \sum_{j=1}^n j \\ &= \sum_{j=1}^n R_{m+j} - n(n+1)/2. \end{aligned}$$

B. The expectation is easy:

$$E(U_{m,n}) = \frac{1}{mn} \sum_{j=1}^m \sum_{i=1}^n P(X_i \leq Y_j) = P(X_1 \leq Y_1) = \int FdG.$$

For the variance, we first calculate

$$\begin{aligned}
E[mnU_{m,n}]^2 &= \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^m \sum_{l=1}^n E1_{[X_i \leq Y_j, X_k \leq Y_l]} \\
&= \sum_{i=1}^m \sum_{j=1}^n E1_{[X_i \leq Y_j]} + \sum_{i \neq k} \sum_{j=1}^n P(X_i \leq Y_j, X_k \leq Y_j) \\
&\quad + \sum_{i=1}^m \sum_{j \neq l} P(X_i \leq Y_j, X_i \leq Y_l) + \sum_{i \neq k} \sum_{j \neq l} P(X_i \leq Y_j, X_k \leq Y_l) \\
&= mnP(X_1 \leq Y_1) + m(m-1)nP(X_1 \leq Y_1, X_2 \leq Y_1) \\
&\quad + mn(n-1)P(X_1 \leq Y_1, X_1 \leq Y_2) \\
&\quad + m(m-1)n(n-1)P(X_1 \leq Y_1, X_2 \leq Y_2),
\end{aligned}$$

$$P(X_1 \leq Y_1) = \int FdG,$$

$$P(X_1 \leq Y_1, X_2 \leq Y_1) = EP(X_1 \leq Y_1, X_2 \leq Y_1 | Y_1) = \int F^2(x)dG(x),$$

$$P(X_1 \leq Y_1, X_1 \leq Y_2) = EP(X_1 \leq Y_1, X_1 \leq Y_2 | X_1) = \int (1 - G(x-))^2 dF(x),$$

and

$$P(X_1 \leq Y_1, X_2 \leq Y_2) = P(X_1 \leq Y_1)^2 = \left(\int FdG \right)^2.$$

It follows by algebra that

$$\begin{aligned}
\text{Var}(mnU_{m,n}) &= E(mnU_{m,n})^2 - \{E(mnU_{m,n})\}^2 \\
&= mn \int FdG + m(m-1)n \int F^2 dG \\
&\quad + mn(n-1) \int (1 - G(x-))^2 dF(x) \\
&\quad + m(m-1)n(n-1) \left\{ \int FdG \right\}^2 - (mn \int FdG)^2 \\
&= m(m-1)n \left\{ \int F^2 dG - \left(\int FdG \right)^2 \right\} \\
&\quad + mn(n-1) \left\{ \int (1 - G(x-))^2 dF(x) - \left(\int FdG \right)^2 \right\} \\
&\quad - mn \int FdG \left(1 - \int FdG \right).
\end{aligned}$$

By noting that

$$\begin{aligned}
\int FdG = P(X \leq Y) &= 1 - P(X > Y) \\
&= 1 - \int G(x-)dF(x) = \int (1 - G(x-))dF(x),
\end{aligned}$$

this yields the claimed variance formula (to within a left limit):

$$\begin{aligned} \text{Var}(\sqrt{mn}U_{m,n}) &= (m-1)\text{Var}(F(Y)) + (n-1)\text{Var}(1-G(X-)) \\ &\quad + \int FdG \left(1 - \int FdG\right). \end{aligned}$$

C. When $F = G$ continuous we find that

$$E(U_{mn}) = \int FdF = 1/2,$$

and, since now $\text{Var}[F(Y)] = \text{Var}[G(X)] = 1/12$,

$$\begin{aligned} \text{Var}(\sqrt{mn}U_{m,n}) &= (m-1)\frac{1}{12} + (n-1)\frac{1}{12} + \frac{1}{4} \\ &= (N-2)\frac{1}{12} + \frac{1}{4} = (N+1)\frac{1}{12}. \end{aligned}$$

Hence from part A it follows that

$$E\left(\sum_{j=1}^n Q_j\right) = n(n+1)/2 + mnE(U_{m,n}) = n(N+1)/2$$

and

$$\text{Var}\left(\sum_{j=1}^n Q_j\right) = mn\text{Var}(\sqrt{mn}U_{m,n}) = mn(N+1)\frac{1}{12}$$

both of which agree with the finite sampling calculations of problem 2.A of problem set 3.

2. Consider the Mann-Whitney-Wilcoxon functional $T(F, G)$ as in problem 1.
 (a) Show that $T(F, G)$ is continuous at every pair of distributions (F, G) with respect to the Kolmogorov distance

$$d_K(F, \tilde{F}) \equiv \sup_x |F(x) - \tilde{F}(x)| \equiv \|F - \tilde{F}\|_\infty :$$

if $\|F_n - F\|_\infty \rightarrow 0$ and $\|G_n - G\|_\infty \rightarrow 0$, then $T(F_n, G_n) \rightarrow T(F, G)$.

(b) Use the result of (a) to prove that $T(\mathbb{F}_n, \mathbb{G}_n) \rightarrow_{a.s.} T(F, G)$.

(c) Give an example to show that $T(F, G)$ is *not* weakly continuous at pairs of distribution functions (F, G) with common discontinuity points.

(d) Extend the definition of Gateaux differentiable functions in a natural way to include $T(F, G)$, and then calculate the Gateaux derivative of $T(F, G)$.

(e) Use your calculation in (d) to “guess” the asymptotic variance of $T(\mathbb{F}_m, \mathbb{G}_n)$.

Solution: 1. (a) $T(F, G)$ is continuous at every pair (F, G) with respect to the Kolmogorov distance: if $\|F_n - F\|_\infty \rightarrow 0$ and $\|G_n - G\|_\infty \rightarrow 0$, then

$$T(F_n, G_n) - T(F, G) = \int (F_n - F)dG_n + \int Fd(G_n - G)$$

where

$$\left| \int (F_n - G_n) dG_n \right| \leq \|F_n - G_n\|_\infty \int dG_n = \|F_n - G_n\|_\infty \rightarrow 0,$$

and, using integration by parts (Proposition 1.4.1, chapter 1, page 17)

$$\left| \int F d(G_n - G) \right| = \left| - \int (G_n(x-) - G(x-)) dF(x) \right| \leq \|G_n - G\|_\infty \rightarrow 0.$$

(b) Since $\|\mathbb{F}_n - F\|_\infty \rightarrow_{a.s.} 0$ and $\|\mathbb{G}_n - G\|_\infty \rightarrow_{a.s.} 0$ by the Glivenko-Cantelli theorem, it follows immediately from the continuity proved in (a) that $T(\mathbb{F}_n, \mathbb{G}_n) \rightarrow_{a.s.} T(F, G)$.

(c) Here is an example to show that $T(F, G)$ is not weakly continuous at all pairs (F, G) : Let $X_n \sim \text{Uniform}(0, 1/n) \equiv F_n$, so that $X_n \rightarrow_d 0 \equiv X \sim \delta_0$, and let $Y_n \sim \text{Uniform}(-1/n, 0) \equiv G_n$ so that $Y_n \rightarrow_d 0 \equiv Y \sim \delta_0$. Then $T(F_n, G_n) = P(X_n \leq Y_n)$ for all n , but $T(F, G) = P(X \leq Y) = 1$. Hence $T(F, G)$ is *not* weakly continuous at *all* (F, G) . However, it is weakly continuous at all pairs (F, G) with *no common discontinuity points*. Here is the proof: suppose that $F_n \rightarrow_d F$ and $G_n \rightarrow_d G$ where F and G have no common discontinuity points. Then $F_n \times G_n \rightarrow_d F \times G$ on $R \times R$: i.e. with $X_n \sim F_n$ and $Y_n \sim G_n$ independent, $(X_n, Y_n) \rightarrow_d (X, Y) \sim F \times G$; here (X, Y) are independent with df's F and G respectively. Since F and G have no common discontinuities, the function $g(x, y) \equiv 1_{[x \leq y]}$ is continuous a.e. $F \times G$: note that all the mass points of the distribution $F \times G$ on R^2 fall off the diagonal, so $P(X = Y) = \int \{F(x) - F(x-)\} dG(x) = 0$. Hence by the Helly-Bray theorem (Proposition 2.3.7, chapter 2, page 13) it follows that

$$T(F_n, G_n) = E1_{[X_n \leq Y_n]} = Eg(X_n, Y_n) \rightarrow Eg(X, Y) = T(F, G).$$

(d) One simple definition of the Gateaux derivative – which does not account for different sample sizes, $m \neq n$, might be as follows: let $F_t \equiv (1-t)F + tF_1$ and $G_t \equiv (1-t)G + tG_1$ for df's F, F_1, G, G_1 . Then

$$(1) \quad \frac{d}{dt} T(F_t, G_t)|_{t=0} = \lim_{t \rightarrow 0} \frac{T(F_t, G_t) - T(F, G)}{t} \equiv \dot{T}(F, G, F_1 - F, G_1 - G).$$

A definition which would account for different sample sizes m and n might be to take $t_m \equiv m^{-1/2}$, $t_n \equiv n^{-1/2}$, and $t_N \equiv (N/mn)^{1/2}$ where $m/N \equiv \lambda_N \rightarrow \lambda \in (0, 1)$, and consider

$$(2) \quad \lim_{m, n \rightarrow \infty} \frac{T(F_{t_m}, G_{t_n}) - T(F, G)}{t_N} \equiv \dot{T}(F, G; \alpha, \beta).$$

We first suppose that $m = n$ and calculate (1): clearly

$$\begin{aligned} T(F_t, G_t) &= \int \{F + t(F_1 - F)\} d\{G + t(G_1 - G)\} \\ &= \int F dG + t \int (F_1 - F) dG + t \int F d(G_1 - G) \\ &\quad + t^2 \int (F_1 - F) d(G_1 - G), \end{aligned}$$

so

$$\begin{aligned}
\frac{d}{dt}T(F_t, G_t)|_{t=0} &= \int (F_1 - F)dG + \int Fd(G_1 - G) \\
&= \int (F_1 - F)dG - \int (G_1 - G)_-dF \\
&= \int G_-d(F_1 - F) + \int Fd(G_1 - G) \\
&= \int (G_- - \int G_-dF)dF_1 + \int (F - \int FdG)dG_1 \\
&= \dot{T}(F, G; F_1 - F, G_1 - G).
\end{aligned}$$

When we evaluate the limit in (2) we obtain a weighted version of this, namely,

$$\sqrt{1 - \lambda} \int (F_1 - F)dG + \sqrt{\lambda} \int Fd(G_1 - G) \equiv \dot{T}(F, G; \alpha, \beta)$$

where $\alpha \equiv \sqrt{1 - \lambda}(F_1 - F)$, $\beta \equiv \sqrt{\lambda}(G_1 - G)$.

3. Consider the collection \mathcal{F}_0 of distribution functions F on R^+ with $0 < E_F X < \infty$ and $E_F X^2 < \infty$. Let $T(F) \equiv \sigma(F)/\mu(F)$ for $F \in \mathcal{F}_0$ where $\sigma^2(F) = Var_F(X)$ and $\mu(F) = E_F(X)$. This is the *coefficient of variation of F* . Find the influence function of $T(F)$.

Solution: $T(F) = \sqrt{\sigma^2(F)}/\mu(F)$ where we already know that

$$\dot{\mu}(F; G - F) = \frac{d}{dt}\mu(F_t)|_{t=0} = \int (x - \int xdF)dG(x)$$

and

$$\dot{\sigma}^2(F; G - F) = \int \{(x - \mu_F)^2 - \sigma_F^2\}dG(x).$$

Thus, by the product rule for differentiation,

$$\dot{T}(F; G - F) = \frac{1}{2} \frac{\sigma(F)}{\mu(F)} \frac{1}{\sigma^2(F)} \dot{\sigma}^2(F; G - F) - \frac{\sigma^2(F)}{\mu^2(F)} \frac{1}{\sigma(F)} \dot{\mu}(F; G - F),$$

and

$$IC(x; T, F) = \frac{1}{2} T(F) \left\{ \left(\frac{x - \mu_F}{\sigma_F} \right)^2 - 1 \right\} - T^2(F) \left(\frac{x - \mu_F}{\sigma_F} \right).$$

4. Suppose that \mathcal{F}_0 is the same class of distribution functions as in problem 3, but now consider the functional $T(F)$ defined for a fixed $x_0 \in R^+$ by

$$T(F) \equiv e_F(x_0) \equiv E_F(X - x_0 | X > x_0) = \frac{\int_{x_0}^{\infty} (1 - F(t))dt}{1 - F(x_0)}.$$

This functional is the *mean residual life* functional. Find the influence function of $T(F)$.

Solution: First note that with $F_t \equiv (1-t)F + tG$ we have both

$$\frac{d}{dt}(1 - F_t(x_0))|_{t=0} = -(G - F)(x_0)$$

and

$$\frac{d}{dt} \int_{x_0}^{\infty} (1 - F_t(y)) dy |_{t=0} = - \int_{x_0}^{\infty} (G - F)(y) dy.$$

Thus by the product rule we calculate

$$\begin{aligned} \frac{d}{dt} T(F_t)|_{t=0} &= - \frac{\int_{x_0}^{\infty} (G - F)(y) dy}{1 - F(x_0)} + \frac{\int_{x_0}^{\infty} (1 - F(y)) dy}{(1 - F(x_0))^2} (G - F)(x_0) \\ &= e_F(x_0) \frac{(G - F)(x_0)}{1 - F(x_0)} - \frac{\int_{x_0}^{\infty} (G - F)(y) dy}{1 - F(x_0)}. \end{aligned}$$

Taking $G = \delta_x = 1_{[x, \infty)}$ yields the influence function for T at F :

$$\begin{aligned} IC(x; T, F) &= e_F(x_0) \frac{(1_{[x, \infty)}(x_0) - F(x_0))}{1 - F(x_0)} - \frac{\int_{x_0}^{\infty} (1_{[x, \infty)}(y) - F(y)) dy}{1 - F(x_0)} \\ &= e_F(x_0) \frac{(1_{[0, x_0]}(x) - F(x_0))}{1 - F(x_0)} - \frac{\int_{x_0}^{\infty} (1_{[0, y]}(x) - F(y)) dy}{1 - F(x_0)} \\ &= \begin{cases} e_F(x_0) - \frac{\int_{x_0}^{\infty} 1 - F(y) dy}{1 - F(x_0)} & x \leq x_0 \\ \frac{-F(x_0)}{1 - F(x_0)} e_F(x_0) - \frac{\int_{x_0}^{\infty} 1_{[0, y]}(x_0) - F(y) dy}{1 - F(x_0)} & x > x_0 \end{cases} \\ &= \begin{cases} 0 & x \leq x_0 \\ \frac{-F(x_0)}{1 - F(x_0)} e_F(x_0) - \frac{\int_{x_0}^x -F(y) dy}{1 - F(x_0)} - \frac{\int_x^{\infty} 1_{[0, y]}(x_0) - F(y) dy}{1 - F(x_0)} & x > x_0 \end{cases} \\ &= \begin{cases} 0 & x \leq x_0 \\ \frac{(x - x_0) - e_F(x_0)}{1 - F(x_0)} & x > x_0 \end{cases} \\ &= \frac{[(x - x_0) - e_F(x_0)] 1_{(x_0, \infty)}(x)}{1 - F(x_0)}. \end{aligned}$$

Note that

$$E_F[IC^2(X; T, F)] = \frac{Var(X - x_0 | X > x_0)}{1 - F(x_0)}.$$

5. Let F be a distribution function on R^2 with finite second moments, and let $\rho(F)$ be the correlation coefficient

$$\rho(F) = \frac{Cov_F(X, Y)}{\sqrt{Var_F(X) Var_F(Y)}}.$$

Find a collection \mathcal{F} of distribution functions on R^2 so that ρ is weakly continuous on \mathcal{F} .

Solution: Consider the following collection of distributions on R^2 : for some $r > 2$ and $M < \infty$

$$\mathcal{F}_{r,M} \equiv \{F : E_F|X|^r \leq M, E_F|Y|^r \leq M\}.$$

Then ρ is weakly-continuous on $\mathcal{F}_{r,M}$ at any F with $Var_F(X) > 0$ and $Var_F(Y) > 0$. Here is a proof: let $\{F_n\} \subset \mathcal{F}_{r,M}$ satisfy $F_n \rightarrow_d F$. Then with $(X_n, Y_n) \sim F_n$ and $(X, Y) \sim F$ we have $(X_n, Y_n) \rightarrow_d (X, Y)$, and by a Skorokhod construction there exist $(X_n^*, Y_n^*) =_d (X_n, Y_n)$ and $(X^*, Y^*) =_d (X, Y)$ defined on a common probability space and satisfying $(X_n^*, Y_n^*) \rightarrow_{a.s.} (X^*, Y^*)$. But because $\{F_n \subset \mathcal{F}_{r,M}, X_n^2, Y_n^2, \text{ and } |X_n Y_n|\}$ are all uniformly integrable: since $r > 2$,

$$EX_n^2 1_{[X_n^2 \geq \lambda]} \leq \frac{1}{\lambda^{r-2}} E|X_n|^r \leq \frac{M}{\lambda^{r-2}}$$

so

$$\lim_{\lambda \rightarrow \infty} \limsup_{n \rightarrow \infty} EX_n^2 1_{[X_n^2 \geq \lambda]} \leq \lim_{\lambda \rightarrow \infty} \frac{M}{\lambda^{r-2}} = 0$$

and similarly for $\{Y_n^2\}$, so the uniform integrability of $|X_n Y_n|$ follows by Cauchy-Schwarz. The same holds true for the (X_n^*, Y_n^*) pairs since the uniform integrability only depends on the (marginal) distributions. Thus by Vitali's theorem it follows that

$$EX_n^s = EX_n^* s \rightarrow EX^* s = EX^s$$

and

$$EY_n^s = EY_n^* s \rightarrow EY^* s = EY^s$$

for $s = 1, 2$, while Vitali also yields

$$EX_n Y_n = EX_n^* Y_n^* \rightarrow EX^* Y^* = EXY.$$

Therefore

$$(3) \quad Var_{F_n}(X_n) \rightarrow Var_F(X), Var_{F_n}(Y_n) \rightarrow Var_F(Y),$$

and

$$(4) \quad Cov_{F_n}(X_n, Y_n) \rightarrow Cov_F(X, Y).$$

Since we have assumed that $Var_F(X) > 0$ and $Var_F(Y) > 0$, (3) and (4) yield

$$\rho(F_n) = \frac{Cov_{F_n}(X_n, Y_n)}{\sqrt{Var_{F_n}(X_n) \cdot Var_{F_n}(Y_n)}} \rightarrow \frac{Cov_F(X, Y)}{\sqrt{Var_F(X) \cdot Var_F(Y)}} = \rho(F);$$

i.e. ρ is continuous on $\mathcal{F}_{r,M}$ at any F with positive variances.

It is interesting to note that the hypothesis $\{F_n\} \subset \mathcal{F}_{r,M}$ cannot be weakened to $\{F_n\} \subset \mathcal{F}_{2,M}$ (and hence it can also not be weakened to the still larger class $\mathcal{F}_{2,\infty}$). Here is a counterexample. Let F be a d.f. on R^2 with $EX = 0 = EY$ and $EX^2 = 1 = EY^2$, and $\rho(F) < 1$ where $(X, Y) \sim F$. Let $M > 1$ be a big number, and consider the class

$$\mathcal{F}_{2,M} = \{F \text{ on } R^2 : E_F X^2 \leq M, E_F Y^2 \leq M\}.$$

Let $a_n, b_n > 0$; we will specify them in terms of M shortly. Consider the sequence of d.f.'s $\{F_n\} \subset \mathcal{F}_{2,M}$ defined by

$$F_n(x, y) = \left(1 - \frac{1}{n}\right)F(x, y) + \frac{1}{2n}\delta_{(a_n, b_n)} + \frac{1}{2n}\delta_{(-a_n, -b_n)}.$$

Then for any bounded and continuous function $\psi : R^2 \rightarrow R$,

$$\begin{aligned} \int \psi dF_n &= \left(1 - \frac{1}{n}\right) \int \psi dF + \frac{1}{2n}\psi(a_n, b_n) + \frac{1}{2n}\psi(-a_n, -b_n) \\ &\rightarrow \int \psi dF, \end{aligned}$$

so $F_n \rightarrow_d F$. Furthermore, with $(X_n, Y_n) \sim F_n$,

$$EX_n = (1 - 1/n)EX = 0, EY_n = 0,$$

$$EX_n^2 = (1 - 1/n)EX^2 + \frac{a_n^2}{n} = (1 - 1/n)M + \frac{a_n^2}{n} = M$$

if $a_n^2 = n\{M - (1 - 1/n)\}$. Similarly,

$$EY_n^2 = (1 - 1/n)EY^2 + \frac{b_n^2}{n} = M$$

if $b_n^2 = n\{M - (1 - 1/n)\}$. With these choices of a_n and b_n ,

$$Cov(X_n, Y_n) = (1 - 1/n)Cov(X, Y) + \frac{a_n b_n}{n},$$

$$\begin{aligned} \rho(F_n) &= \frac{Cov(X_n, Y_n)}{\sqrt{Var(X_n)Var(Y_n)}} \\ &= \frac{(1 - 1/n)Cov(X, Y) + M - (1 - 1/n)}{\sqrt{M^2}} \\ &\rightarrow \frac{rho(F) + M - 1}{M} \neq \rho(F). \end{aligned}$$

Thus $\rho(F)$ is not continuous on $\mathcal{F}_{2,M}$.

6. For distribution functions F on R^+ and $t_0 > 0$, consider the functional

$$T(F) = \Lambda(t_0) \equiv \int_0^{t_0} \frac{1}{1 - F_-} dF,$$

the cumulative hazard function corresponding to F at t_0 . Find the influence function of $T(F)$. What does this mean about asymptotic normality of the natural estimator $T(\mathbb{F}_n)$ of $T(F)$? Can you prove asymptotic normality of $T(\mathbb{F}_n)$ directly?

Solution: To find the influence function of $T(F)$, let $F_t = (1 - t)F + t\delta_x$. The distribution function corresponding to $G \equiv \delta_x$ is $1_{(-\infty, y]}(x)$, $y \in R$, so the left limit

is $G_-(y) = 1_{[x < y]}$, and the corresponding "at risk" function $1 - G_-(y) = 1_{[x \geq y]} = 1_{[y, \infty)}(x)$. We need to compute

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{T(F_t) - T(F)}{t} &= \frac{d}{dt} T(F_t)|_{t=0} \equiv IC(x; T, F) \equiv \psi_F(x) \\
&= \frac{d}{dt} \left\{ \int_0^{t_0} \frac{1}{1 - (F_t)_-} dF_t \right\} |_{t=0} \\
&= \int_0^{t_0} \frac{1}{1 - F_-} d(\delta_x - F) + \int_0^{t_0} \frac{(\delta_x - F_-)}{(1 - F_-)^2} dF \\
&= \frac{1_{[0, t_0]}(x)}{1 - F_-(x)} - \int_0^{t_0} \frac{1}{1 - F_-} dF + \int_0^{t_0} \frac{1}{1 - F_-} dF - \int_0^{t_0} \frac{(1 - \delta_{x-})}{(1 - F_-)^2} dF \\
&= \frac{1_{[0, t_0]}(x)}{1 - F_-(x)} - \int_0^{t_0} \frac{1_{[x \geq y]}}{(1 - F_-(y))^2} dF(y) \\
&= \frac{1_{[0, t_0]}(x)}{1 - F_-(x)} - \int_0^x \frac{1_{[0, t_0]}(y)}{1 - F_-(y)} d\Lambda(y) \\
&= \begin{cases} \frac{1}{1 - F_-(x)} - \int_0^x \frac{1}{(1 - F_-)^2} dF & \text{if } x \leq t_0 \\ - \int_0^{t_0} \frac{1}{(1 - F_-)^2} dF & \text{if } x > t_0. \end{cases}
\end{aligned}$$

The next to last formula for the influence function of $\Lambda(t_0)$ is natural from a martingale perspective. When F is continuous $F_- = F$, and the influence function computed above reduces to:

$$\begin{aligned}
IC(x; T, F) &= 1_{[x \leq t_0]} - \frac{F(t_0)}{1 - F(t_0)} 1_{[x > t_0]} \\
&= \frac{1_{[x \leq t_0]} - F(t_0)}{1 - F(t_0)}.
\end{aligned}$$

Note that $E_F \psi_F(X) = 0$ and (in the case of a continuous d.f. F)

$$E_F \psi_F^2(X) = \frac{F(t_0)}{1 - F(t_0)}.$$

To prove asymptotic normality of $T(\mathbb{F}_n)$ (assuming that F satisfies $F(t_0) < 1$), write

$$\begin{aligned}
\sqrt{n}(T(\mathbb{F}_n) - T(F)) &= \sqrt{n} \left\{ \int_0^{t_0} \frac{1}{1 - \mathbb{F}_n(s-)} d\mathbb{F}_n(s) - \int_0^{t_0} \frac{1}{1 - F(s-)} dF(s) \right\} \\
&= \int_0^{t_0} \frac{1}{1 - \mathbb{F}_n(s-)} d[\sqrt{n}(\mathbb{F}_n(s) - F(s))] \\
&\quad + \int_0^{t_0} \sqrt{n} \left\{ \frac{1}{1 - \mathbb{F}_n(s-)} - \frac{1}{1 - F(s-)} \right\} dF(s) \\
&= \frac{\sqrt{n}(\mathbb{F}_n(t_0) - F(t_0))}{1 - \mathbb{F}_n(t_0-)} - \int_0^{t_0} \sqrt{n}(\mathbb{F}_n(s) - F(s))^2 \frac{1}{(1 - \mathbb{F}_n(s-))^2} d\mathbb{F}_n(s) \\
&\quad + \int_0^{t_0} \frac{\sqrt{n}(\mathbb{F}_n(s) - F(s))}{(1 - \mathbb{F}_n(s-))(1 - F(s-))} dF(s) \\
&= \frac{\sqrt{n}(\mathbb{F}_n(t_0) - F(t_0))}{1 - \mathbb{F}_n(t_0-)} + o_p(1)
\end{aligned}$$

$$\begin{aligned}
&\xrightarrow{d} \frac{\mathbb{U}(F(t_0))}{1 - F(t_0-)} \\
&\sim N\left(0, \frac{F(t_0)}{1 - F(t_0)}\right) \quad \text{if } F \text{ is continuous}
\end{aligned}$$

since the last two terms can be rewritten as

$$\int_0^{t_0} \frac{\sqrt{n}(\mathbb{F}_n(s) - F(s))}{1 - \mathbb{F}_n(s-)} \left\{ \frac{d\mathbb{F}_n(s)}{1 - \mathbb{F}_n(s-)} - \frac{dF(s)}{1 - F(s-)} \right\} = o_p(1)$$

by arguments similar to those we used to deal with the Mann-Whitney Wilcoxon statistic. Alternatively, martingale methods also work.