Statistics 583, Problem Set 4 Solutions

Wellner; 4/26/2000

1. Suppose that $X_1, \ldots, X_m$ are i.i.d. $F \in \mathcal{F}_c$ and $Y_1, \ldots, Y_n$ are i.i.d. $G \in \mathcal{F}_c$.
A. Find a most powerful similar test of $H_c : F = G \in \mathcal{F}_c$ versus

$$K_1 : F = \text{Exponential}(\mu), \quad G = \text{Exponential}(\nu) \quad \text{with} \quad \nu < \mu \quad \text{known.}$$

B. Does the resulting test have any optimality properties against any composite alternative?
C. If $m = n = 4$ and we observe $\underline{X} = (X_1, X_2, X_3, X_4) = (2.61, 3.02, 1.97, 2.79)$ and $\underline{Y} = (Y_1, Y_2, Y_3, Y_4) = (3.28, 2.19, 2.88, 3.41)$, carry out the test in A at level $\alpha = .1$. What is the approximate $p-$value for the observed data?

Solution: A. For testing H_c versus K_1, the joint density under the alternative is

$$h(\underline{x}, \underline{y}) = \mu^m \nu^n \exp(-\mu \sum_1^m x_i - \nu \sum_1^n y_j) .$$

The most powerful similar test rejects H_c for those permutations $\underline{z}'$ of $\underline{z}$ which lead to large values of $h(\underline{z})$; or small values of $\mu \sum_1^m X_i + \nu \sum_1^n Y_j$; or small values of

$$\begin{aligned} &\mu \sum_1^m X_i + \nu \sum_1^n Y_j - (\frac{m}{N}\mu + \frac{n}{N}\nu)(\sum_1^m X_i + \sum_1^n Y_j) \\ &= \frac{mn}{N}(\mu - \nu)(\overline{X}_m - \overline{Y}_n) ; \end{aligned}$$

or large values of $(\overline{Y}_n - \overline{X}_m)$; or small values of

$$\mu \sum_1^m X_i + \nu \sum_1^n Y_j - \mu(\sum_1^m X_i + \sum_1^n Y_j) = (\nu - \mu) \sum_1^n Y_j ;$$

or large values of $\sum_1^n Y_j$.
B. Since the test in A is the same for all $\nu < \mu$, it is actually a UMP similar test of H_c versus $\cup_{\nu<\mu} K_1(\mu, \nu)$.
C. The ordered data is given by

$$\underline{Z} = (1.97, 2.19, 2.61, 2.79, 2.88, 3.02, 3.28, 3.41) .$$

There are $\binom{8}{4} = 70$ relabelings of the data as $X's$ and $Y's$. Carrying out the permutation test using $\sum_1^n Y_j$, we easily find (I used Mathematica; see the attached data) that for the observed data, $\sum_1^n Y_j = 11.76$, and there are 13 combinations with sums greater than or equal to 11.76; hence the $p-$value for the permutation test is $13/70 = .1857$. Thus we fail to reject at level $\alpha = .1$.

2. Consider the testing problem in Problem 1. A. In the context of Problem 1, consider the smaller parametric null hypothesis

$$H_0: \ F = \text{Exponential}(\mu), \, G = \text{Exponential}(\mu)\,.$$

Find the UMP unbiased test of H_0 versus $K_0 : \nu < \mu$. (This is closely related to problem 2 of problem set #3.)
B. Can you rewrite the UMPU test from part A in such a way that its critical region is asymptotically equivalent to the critical region of the permutation test you found in problem 1? [Hint: you will need to use the WWNH finite sampling CLT.]

Solution: A. For testing H_0 versus K_0, the UMPU test is "reject H_0 if $V \equiv \sum_1^n Y_j / (\sum_1^m X_i + \sum_1^n Y_j) > b_{n,m,\alpha}$" where $b_{n,m,\alpha}$ is the upper $\alpha-$th percentile of the Beta(n, m) distribution. [Equivalently, since V is a monotone increasing function of $R \equiv \sum_1^n Y_j / \sum_1^m X_i$, and $(m/n)R \sim F_{2n,2m}$, the test is "reject if $(m/n)R > F_{2n,2m,\alpha}$".] Under H_0 we have $E(V) = n/N$ and $Var(V) = \lambda_N \overline{\lambda}_N/(N+1)$, where $\lambda_N \equiv m/N$, $\overline{\lambda}_N \equiv 1 - \lambda_N$, and

$$\text{(1)} \qquad \frac{V - n/N}{\sqrt{\lambda_N \overline{\lambda}_N/(N+1)}} \to_d N(0,1)\,.$$

Hence the quantiles $b_{n,m,\alpha}$ satisfy

$$\frac{b_{n,m,\alpha} - n/N}{\sqrt{\lambda_N \overline{\lambda}_N/(N+1)}} \to z_\alpha\,.$$

Proof of (1): Note that

$$\begin{aligned} V - \frac{n}{N} &= \frac{mn}{N^2}(\overline{Y}_n - \overline{X}_m)\frac{N}{\sum_1^m X_i + \sum_1^n Y_j} \\ &= \lambda_N \overline{\lambda}_N(\overline{Y}_n - \mu - (\overline{X}_m - \mu))\frac{1}{\lambda_N \overline{X}_m + \overline{\lambda}_N \overline{Y}_n}\,. \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{V - n/N}{\sqrt{\lambda_N \overline{\lambda}_N/(N+1)}} \\ &= \sqrt{\lambda_N \overline{\lambda}_N}\sqrt{N+1}(\overline{Y}_n - \mu - (\overline{X}_m - \mu))\frac{1}{\lambda_N \overline{X}_m + \overline{\lambda}_N \overline{Y}_n} \\ &= \sqrt{\frac{N+1}{N}}\left(\sqrt{\overline{\lambda}_N}\sqrt{m}(\overline{Y}_n - \mu) - \sqrt{\lambda_N}\sqrt{n}(\overline{X}_m - \mu)\right)\frac{1}{\lambda_N \overline{X}_m + \overline{\lambda}_N \overline{Y}_n} \\ &\to_d \left(\sqrt{\overline{\lambda}} Z - \sqrt{\lambda} W\right)/\mu \\ &\qquad \text{where } Z \sim N(0,\mu^2), W \sim N(0,\mu^2) \ \text{ are independent} \\ &\sim N(0,1)\,. \end{aligned}$$

B. The permutation test in problem 1 rejects for large values of $\sum_1^n Y_j$, or equivalently for large values of

$$V = \frac{\sum_1^n Y_j}{\sum_1^N Z_i} = \frac{n}{N}\frac{\overline{Y}_n}{\overline{Z}_N}.$$

Now $E(\overline{Y}_n|\underline{Z}) = \overline{Z}$, so $E(V|\underline{Z}) = n/N$. Furthermore,

$$\begin{aligned} Var(V|\underline{Z}) &= \overline{\lambda}_N^2 Var(\overline{Y}_n|\underline{Z})/\overline{Z}_N^2 \\ &= \overline{\lambda}_N^2 \frac{\sigma_N^2}{n}(1 - \frac{n-1}{N-1})\frac{1}{\overline{Z}^2}. \end{aligned}$$

Thus we have

$$\frac{V - E(V|\underline{Z})}{\sqrt{Var(V|\underline{Z})}} = \frac{\overline{Y}_n - \overline{Z}_N}{\frac{\sigma_N}{\sqrt{n}}\sqrt{1 - \frac{n-1}{N-1}}} \to_d N(0,1)$$

almost surely (with respect to the $\underline{Z}'s$) by the W-W-N-H finite-sampling CLT once we verify the Noether condition. If the X's and Y's have finite second moments, this holds in exactly the same way as it did for the permutation $t-$ test. Note that

$$\begin{aligned} \frac{V - E(V|\underline{Z})}{\sqrt{Var(V|\underline{Z})}} &= \frac{V - n/N}{\sqrt{\lambda_N\overline{\lambda}_N/(N+1)}} \cdot \sqrt{\frac{\lambda_N\overline{\lambda}_N}{N+1}} \frac{\overline{Z}_N}{\overline{\lambda}_N \frac{\sigma_N}{\sqrt{n}}\sqrt{1 - \frac{n-1}{N-1}}} \\ &= \frac{V - n/N}{\sqrt{\lambda_N\overline{\lambda}_N/(N+1)}} \cdot \frac{\overline{Z}_N}{\sigma_N} \cdot (1 + o(1)), \end{aligned}$$

where $\overline{Z}_N/\sigma_N \to_p 1$ under the exponential parametric model.

3. Consider the critical point $c_\alpha(\underline{Z})$ of the two-sample permutation $t-$test. In class on 4/17 and 4/21 we showed that $c_\alpha(\underline{Z}) \to_p z_\alpha$ under both the null hypothesis $F = G$ and under the alternative hypothesis $F \neq G$ if $E_F X^2 < \infty$ and $E_G Y^2 < \infty$. What does this imply about the power of the permutation test as $m \wedge n \to \infty$ if $E_F X < E_G Y$?

   Solution: When $\mu_X \equiv E_F X < E_G Y \equiv \mu_Y$, we have

$$\begin{aligned} \tau_{m,n} &= \frac{\sqrt{\frac{mn}{N}}(\overline{Y}_n - \overline{X}_m)}{S_p} \\ &= \frac{\sqrt{\frac{mn}{N}}((\overline{Y}_n - \mu_Y - (\overline{X}_m - \mu_X)}{S_p} \\ &\quad + \frac{\sqrt{\frac{mn}{N}}(\mu_Y - \mu_X)}{S_p} \\ &\to_p N(0,1) + \infty = \infty. \end{aligned}$$

   Since we already know that $c_\alpha(\underline{Z}) \to_p z_\alpha$ under alternatives with $E_F X^2 < \infty$ and $E_G Y^2 < \infty$, it follows that

$$P_{\mu_X < \mu_Y}(\tau_{m,n} \geq c_\alpha(\underline{Z})) \to 1.$$

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$ and consider testing $H : |\mu| \ge 1$ versus $K : |\mu| < 1$. This is the "bioequivalence" problem considered by Perlman and Wu (2000). See (1) on page 356 of Perlman and Wu, and Section 7, pages 361 - 362. Find the likelihood ratio test of H versus K. How is your rejection region related to the set R in (22) of Perlman and Wu?

Solution: The likelihood is

$$\begin{aligned} L_n(\mu, \sigma^2) &= (2\pi\sigma^2)^{-1/2} \exp\left(- \frac{\sum_1^n (X_i - \mu)^2}{2\sigma^2} \right) \\ &= (2\pi\sigma^2)^{-1/2} \exp\left(- \frac{n}{2\sigma^2} (S_X^2 + (\overline{X}_n - \mu)^2) \right) \end{aligned}$$

where $\overline{X}_n \equiv n^{-1} \sum_{i=1}^n X_i$ and $S_X^2 = n^{-1} \sum_{i=1}^n (X_i - \overline{X})^2$. The unrestricted maximum is easy and well-known:

$$\sup_{\Theta} L_n(\mu, \sigma^2) = (2\pi S_X^2)^{-n/2} \exp(-n/2) \,,$$

and the supremum is attained at $\hat{\mu} \equiv \overline{X}_n$, $\hat{\sigma}^2 = S_X^2$. To maximize the likelihood over the null hypothesis $\Theta_0 \equiv \{(\mu, \sigma) : |\mu| \ge 1, \sigma^2 > 0\}$, we first maximize over μ for each fixed σ^2: by considering the shape of the quadratic function $(\mu - \overline{X}_n)^2$ in the three cases $\overline{X}_n < -1$, $-1 \le \overline{X}_n \le 1$, and $\overline{X}_n > 1$, it becomes clear that the maximum likelihood estimator $\hat{\mu}_0$ under the null hypothesis $|\mu| \ge 1$ is given by

$$\hat{\mu}_0 = \begin{cases} \overline{X}_n & \text{if} \quad |\overline{X}_n| \ge 1 \\ 1 & \text{if} \quad 0 < \overline{X}_n < 1 \\ -1 & \text{if} \quad -1 < \overline{X}_n < 0 \,. \end{cases} ,$$

and this does not depend on the value of σ^2. Hence the partially maximized log-likelihood is given by

$$\log L_n(\hat{\mu}_0, \sigma^2) = \begin{cases} -\frac{n}{2} \left(\log(2\pi\sigma^2) - \frac{S_X^2}{\sigma^2} \right) & \text{if} \quad |\overline{X}_n| \ge 1 \\ -\frac{n}{2} \left(\log(2\pi\sigma^2) - \frac{S_X^2 + (\overline{X}_n - 1)^2}{\sigma^2} \right) & \text{if} \quad 0 < \overline{X}_n < 1 \\ -\frac{n}{2} \left(\log(2\pi\sigma^2) - \frac{S_X^2 + (\overline{X}_n + 1)^2}{\sigma^2} \right) & \text{if} \quad -1 < \overline{X}_n < 0 \,. \end{cases}$$

Maximizing this with respect to σ^2 is easy, and we find that

$$\hat{\sigma}_0^2 = \begin{cases} S_X^2 & \text{if} \quad |\overline{X}_n| \ge 1 \\ S_X^2 + (\overline{X}_n - 1)^2 & \text{if} \quad 0 < \overline{X}_n < 1 \\ S_X^2 + (\overline{X}_n + 1)^2 & \text{if} \quad -1 < \overline{X}_n < 0 \,. \end{cases}$$

Hence we find that the likelihood ratio statistic for testing H versus K is given by

$$\begin{aligned} \lambda_n &\equiv \frac{\sup_{\Theta} p(\underline{X}, \mu, \sigma^2)}{\sup_{\Theta_0} p(\underline{X}, \mu, \sigma^2)} \\ &= \begin{cases} \frac{(2\pi S_X^2)^{-n/2} \exp(-n/2)}{(2\pi S_X^2)^{-n/2} \exp(-n/2)} & \text{if} \quad |\overline{X}_n| \ge 1 \\ \frac{(2\pi S_X^2)^{-n/2} \exp(-n/2)}{(2\pi S_X^2 + (\overline{X}_n - 1)^2)^{-n/2} \exp(-n/2)} & \text{if} \quad 0 < \overline{X}_n < 1 \\ \frac{(2\pi S_X^2)^{-n/2} \exp(-n/2)}{(2\pi S_X^2 + (\overline{X}_n + 1)^2)^{-n/2} \exp(-n/2)} & \text{if} \quad -1 < \overline{X}_n < 0 \end{cases} \end{aligned}$$

$$= \begin{cases} 1 & \text{if} \quad |\overline{X}_n| \geq 1 \\ \left(\frac{S_X^2 + (\overline{X}_n - 1)^2}{S_X^2}\right)^{n/2} & \text{if} \quad 0 < \overline{X}_n < 1 \\ \left(\frac{S_X^2 + (\overline{X}_n + 1)^2}{S_X^2}\right)^{n/2} & \text{if} \quad -1 < \overline{X}_n < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if} \quad |\overline{X}_n| \geq 1 \\ \left(1 + \left(\frac{\overline{X}_n - 1}{S_X}\right)^2\right)^{n/2} & \text{if} \quad 0 < \overline{X}_n < 1 \\ \left(1 + \left(\frac{\overline{X}_n + 1}{S_X}\right)^2\right)^{n/2} & \text{if} \quad -1 < \overline{X}_n < 0\,. \end{cases}$$

Thus large values of the likelihood ratio λ_n correspond to large values of $(1 - \overline{X}_n)/S_X$ for $0 \leq \overline{X}_n < 1$, and large values of $(\overline{X}_n + 1)/S_X$ for $-1 < \overline{X}_n < 0$. Thus we reject for values of $(\overline{X}_n, S_X)$ in the triangular region

$$\begin{aligned} R &\equiv \{(\overline{X}_n, S_X) : \ \overline{X}_n + c_\alpha S_X \leq 1, \ 0 < \overline{X}_n < 1\} \\ &\qquad \cup \{(\overline{X}_n, S_X) : \ c_\alpha S_X - \overline{X}_n \leq 1, \ -1 < \overline{X}_n < 0\}\} \\ &= \{(\overline{X}_n, S_X) : |\overline{X}_n| + c_\alpha S_X \leq 1\}\,. \end{aligned}$$

To determine c_α so that the resulting test has size α, we will combine two one-sided $t-$tests. Note that when $\mu = 1$, and with $\tilde{S}_n^2 \equiv (n/(n-1))S_X^2$,

$$\sqrt{n-1}\frac{\overline{X}_n - 1}{S_X} = \frac{\sqrt{n}(\overline{X}_n - 1)}{\tilde{S}_X} \sim t_{n-1}\,.$$

Thus the one-sided t-test of $H_1 : \mu \geq 1$ versus $K_1 : \mu < 1$ is "reject H_1 at level α if

$$\sqrt{n-1}\frac{\overline{X}_n - 1}{S_X} = \frac{\sqrt{n}(\overline{X}_n - 1)}{\tilde{S}_X} < -t_{n-1,\alpha}".$$

Note that this corresponds to pairs $(\overline{X}_n, S_X)$ in the region R_1 consisting of all points below the line $\{(x,s) : x + (t_{n-1,\alpha}/\sqrt{n-1})s = 1\}$. Call this test Ψ_1.

Note that when $\mu = -1$, and with $\tilde{S}_n^2 \equiv (n/(n-1))S_X^2$,

$$\sqrt{n-1}\frac{\overline{X}_n + 1}{S_X} = \frac{\sqrt{n}(\overline{X}_n + 1)}{\tilde{S}_X} \sim t_{n-1}\,.$$

Thus the (UMPU) one-sided t-test of $H_2 : \mu \leq -1$ versus $K_2 : \mu > -1$ is "reject H_2 at level α if

$$\sqrt{n-1}\frac{\overline{X}_n + 1}{S_X} = \frac{\sqrt{n}(\overline{X}_n + 1)}{\tilde{S}_X} > t_{n-1,\alpha}".$$

Note that this corresponds to pairs $(\overline{X}_n, S_X)$ in the region R_2 consisting of all points below the line $\{(x,s) : -x + (t_{n-1,\alpha}/\sqrt{n-1})s = 1\}$. Call this test Ψ_2.

Theorem (Berger (1982), Berger and Hsu (1996)): the test which rejects $H : [\mu \leq -1] \cup [\mu \geq 1] \equiv \Theta_{01} \cup \Theta_{02} \equiv \Theta_0$ in favor of $K : [\mu > -1] \cap [\mu < 1] \equiv \Theta_{01}^c \cap \Theta_{02}^c$ for pairs $(\overline{X}_n, S_X)$ in $R = R_1 \cap R_2$ has size α. Thus with $c_\alpha = t_{n-1,\alpha}/\sqrt{n-1}$, the test

$\Psi(\underline{X}) = 1_R(\underline{X})$ has size α.
Proof: Suppose that $\theta = (\mu, \sigma^2) \in \Theta_0$. Then $\theta \in \Theta_{0i}$ for some $i \in \{1, 2\}$, and

$$E_\theta \Psi(\underline{X}) \le E_\theta 1_{R_i}(\underline{X}) \le \alpha$$

since the test $\Psi_i = 1\{R_i\}$ has size α. Thus $\sup_{\theta \in \Theta_0} E_\theta \Psi \le \alpha$. Moreover, for the points $\theta_i = (1, 1/i) \to (1, 0)$ we have $\theta_i \in \Theta_0$ for all i and

$$E_{\theta_i} \Psi_1 = \alpha \qquad \text{for all } i$$

while

$$E_{\theta_i} \Psi_2 = P_{\theta_i}(\sqrt{n}(\overline{X}_n + 1)/\tilde{S}_X > t_{n-1,\alpha}) \to 1 \qquad \text{as } i \to \infty\,.$$

Thus

$$\begin{aligned}
\sup_{\theta \in \Theta_0} E_\theta 1_R(\underline{X}) &\ge \lim_i E_{\theta_i} 1_R(\underline{X}) = \lim_i P_{\theta_i}(R_1 \cap R_2) \\
&= \lim_i [1 - P_{\theta_i}(R_1^c \cup R_2^c) \ge \lim_i [1 - P_{\theta_i}(R_1^c) - P_{\theta_i}(R_2^c)] \\
&= 1 - (1 - \alpha) - 0 = \alpha\,.
\end{aligned}$$

Thus $\sup_{\theta \in \Theta_0} E_\theta \Psi = \alpha$.

5. Confidence intervals can be obtained by "inverting" tests, If a test has an optimality property such as being the UMP unbiased test of $H: \theta = \theta_0$ versus $K: \theta \ne \theta_0$, then the corresponding confidence sets often have a corresponding optimality property. Furthermore, note that a family of confidence sets with confidence level decreasing to zero will define an estimator. See Lehmann, TSH, sections 3.5 and 5.7. The following problem is in this vein. Suppose that $X \sim \text{Binomial}(m, p_1)$ and $Y \sim \text{Binomial}(n, p_2)$. Find the most accurate unbiased confidence interval for the the log-odds ratio

$$\theta \equiv \log\left(\frac{p_2/q_2}{p_1/q_1}\right)\,.$$

[Hint: See Lehmann, TSH, section 5.7, page 221, for a related example involving comparison of two Poisson parameters.]

Solution: This is a less specific version of Problem 32, Chapter 5, page 261, Lehmann, TSH. We saw in class that the conditional distribution of Y given $T = X + Y = t$ is given by

$$P(Y = y | X + Y = t) = C_t(\rho) \binom{m}{t-y} \binom{n}{y} \rho^y$$

for $y = 0, \ldots, t$ where $\rho = (p_2/q_2)/(p_1/q_1) = e^\theta$ and $C_t(\rho)$ is a norming constant. As discussed in Lehmann page 221, the UMP unbiased tests of $\theta = \theta_0$ are performed conditionally given $T = t$, and the confidence intervals for θ will, as a consequence, also be obtained conditionally. For continuous conditional distributions the acceptance regions will be of the form

$$C_1(\theta, t) \le Y \le C_2(\theta, t),$$

where, for each t the functions $C_i(\theta, t)$ are increasing by Lemma 2, Lehmann, TSH. The confidence intervals are then

$$C_2^{-1}(Y; t) \leq \theta \leq C_1^{-1}(Y; t).$$

In the present integer-valued case, we can apply Lehmann's Lemma 2 to $Y + V$ where $V \sim \text{Uniform}(0, 1)$ is independent of both X and Y. The upshot is that the confidence intervals are of the form

$$C_2^{-1}(Y + V; t) \leq \theta \leq C_1^{-1}(Y + V; t).$$

where $C_i(\theta, t)$ are chosen to satisfy

$$E_\theta(1_{[C_1(\theta,t), C_2(\theta,t)]}(Y + V)|T = t) = 1 - \alpha$$

and

$$E_\theta((Y + V)[1 - 1_{[C_1(\theta,t), C_2(\theta,t)]}(Y + V)]|T = t) = \alpha E_\theta(Y + V|T = t)\,.$$

As you can see, these are somewhat painful intervals to find. Lehmann's problem asks only for existence of the intervals. But they have been implemented – and indeed several software companies have been formed to compute these and other intervals of the same type; see e.g. Liao and Hall (1995), Coe and Tamhane (1993), and Granville and Schifflers (1993).

References:

Granville, V. and Schifflers, E. (1993). Efficient algorithms for exact inference in 2x2 contingency tables. *Statistics and Computing* **3**, 83-87.

Coe, Paul R. and Tamhane, A. C. (1993). Small sample confidence intervals for the difference, ratio, and odds ratio of two success probabilities. *Commun. Statist. - Simul* **22**, 925-938.

Liao, J. G. and Hall, C. B. (1995). Fast and accurate computation of the exact confidence limits for the common odds ratio in several 2x2 tables. *J. Computational Graphical Statistics* **4**, 173 - 179.