Statistics 583, Problem Set 3 Solutions

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1. Let $X \sim$ Exponential(μ) and $Y \sim$ Exponential(ν) be independent random variables: thus the joint density of (X, Y) is

$$p(x, y; \mu, \nu) = \mu\nu \exp(-\mu x - \nu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y) \, .$$

Find a UMP unbiased test of size $\alpha = .2$ for testing:
(a) $H_0 : \mu \le \nu + 1$ against $H_1 : \mu > \nu + 1$.
(b) $H_0 : \mu = \nu$ against $H_1 : \mu \ne \nu$.
(c) $H_0 : \mu \ge 2\nu$ against $H_1 : \mu < 2\nu$.

Solution: When $X \sim$ Exponential(μ) and $Y \sim$ Exponential(ν), we have

$$p_{\mu,\nu}(x, y) = \mu\nu \exp(-\mu x - \nu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y) \, .$$

(a) For testing $H : \mu \le \nu + 1$ versus $\mu > \nu + 1$, we rewrite the density as follows:

$$\begin{aligned} p_{\mu,\nu}(x, y) &= \mu\nu \exp(-\mu x - \nu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y) \\ &= \mu\nu \exp((\mu - \nu) y - \mu(x + y)) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y) \\ &= \mu\nu \exp(\theta U(x, y) + \xi T(x, y)) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y) \end{aligned}$$

where $\theta \equiv \mu - \nu$, $U(x, y) \equiv y$, $\xi \equiv -\mu$, and $T(x, y) \equiv x + y$. Since $\mu \le \nu + 1$ is equivalent to $\mu - \nu = \theta \le 1 \equiv \theta_0$, our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X, Y) = \begin{cases} 1 & \text{if } Y > c_\alpha(T) \\ \gamma(T) & \text{if } Y = c_\alpha(T) \\ 1 & \text{if } Y < c_\alpha(T) \end{cases}$$

where c_α and γ_α satisfy $E\{\phi(X, Y)|T\} = \alpha$. In this case, the conditional distribution of Y given $T = X + Y$ on the boundary $\Theta_B = \{(\mu - 1, \mu) : \mu \ge 1\}$ is given by

$$f_{Y|T}(y|t) = \frac{e^y}{e^t - 1} 1_{[0,t]}(y) \, .$$

Therefore $1 - F_{Y|T}(y|t) = 1 - (e^y - 1)/(e^t - 1)$, and for $\alpha = .2$ the critical point for the conditional test is given by

$$c_{\alpha T} = \log\{e^T - (e^T - 1)/5\} = \log\{(4/5)e^T + 1/5\}, \qquad \gamma(T) = 0 \, .$$

(b) For testing $H : \mu = \nu$ versus $K : \mu \ne \nu$, the same rewrite of the density as in (a) works. Now we have $\mu = \nu$ is equivalent to $\mu - \nu = 0 \equiv \theta_0$, and $\mu \ne \nu$ is equivalent to $\mu - \nu \neq 0 \equiv \theta_0$, so our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X,Y) = \begin{cases} 1 & \text{if } Y > c_2(T) \text{ or } Y < c_1(T) \\ \gamma_i(T) & \text{if } Y = c_i(T),\ i = 1,2 \\ 0 & \text{if } c_1(T) < Y < c_2(T) \end{cases}$$

where the c_1, c_2, γ_1, and γ_2 are determined so that $E\{\phi(X,Y)|T\} = \alpha$ almost surely. In this case the conditional distribution of Y given T on $\Theta_B = \{(\mu,\mu) : \mu \geq 0\}$ is Uniform$(0,T)$, and hence the conditional distribution of Y/T given T is Uniform$(0,1)$, and this is independent of T. Hence the UMPU test of H versus K of size .2 is given by "reject H if $Y/T < .1$ or $Y/T > .9$".
(c) For testing $H : \mu \geq 2\nu$ versus $K : \mu < 2\nu$, we need a somewhat different rewrite of the joint density:

$$\begin{aligned} p_{\mu,\nu}(x,y) &= \mu\nu\exp(-\mu x - \nu y) \\ &= \mu\nu\exp(-(\mu - 2\nu)x - \nu(2x+y)) \\ &= \mu\nu\exp(\theta U(x,y) + \xi T(x,y)) \end{aligned}$$

where $\theta \equiv 2\nu - \mu$, $U(x,y) \equiv x$, $\xi \equiv -\nu$, and $T(x,y) \equiv 2x + y$. Since $\mu \geq 2\nu$ is equivalent to $2\nu - \mu \equiv \theta \leq 0 \equiv \theta_0$ (and $\mu < 2\nu$ is equivalent to $2\nu - \mu = \theta > 0 \equiv \theta_0$), our theory for exponential families applies, and the UMP unbiased test of H versus K is given by

$$\phi(X,Y) = \begin{cases} 1 & \text{if } X > c_\alpha(T) \\ \gamma(T) & \text{if } X = c_\alpha(T) \\ 1 & \text{if } X < c_\alpha(T) \end{cases}$$

where $c_\alpha(T)$ and $\gamma(T)$ satisfy $E\{\phi(X,Y)|T\} = \alpha$. In this case the conditional distribution of X given T under $\theta \in \Theta_B = \{(2\nu,\nu) : \nu \geq 0\}$ is Uniform$(0,T/2)$, so $2X/T$ is Uniform$(0,1)$ and independent of T. Hence the UMPU test of H versus K of size $\alpha = .2$ is given by "reject H if $2X/T > .8$".

2. Suppose that we change problem 1 as follows: $X_1,\ldots,X_m$ are i.i.d. Exponential(μ), and $Y_1,\ldots,Y_n$ are i.i.d. Exponential(ν).
A. Find a UMP unbiased test of the hypotheses in (b) in this case.
B. Do the methods of problem 1 work for hypotheses of the form $H : g(\mu,\nu) \leq 0$ versus $K : g(\mu,\nu) > 0$ and $g_1(\mu,\nu) = a\mu + b\nu + c$ for constants a,b,c? What about $g_2(\mu,\nu) = a(\mu/\nu) - b$ or $g_3(\mu,\nu) = a\mu\nu - b$?

Solution: A. When we observe $X_1,\ldots,X_m$ i.i.d. Exponential(μ) and $Y_1,\ldots,Y_n$ i.i.d. Exponential(ν), then the distribution of the data is given by the joint density

$$\begin{aligned} p_{\mu,\nu}(\underline{x},\underline{y}) &= \mu^m\nu^n\exp(-\mu\sum_{i=1}^m x_i - \nu - \nu\sum_{i=1}^n y_i) \\ &= \mu^n\nu^n\exp((\mu-\nu)\sum_{j=1}^n y_j - \mu(\sum_{i=1}^m x_i + \sum_{j=1}^n y_j)) \\ &= \mu^n\nu^n\exp(\theta U(\underline{x},\underline{y}) + \xi T(\underline{x},\underline{y})) \end{aligned}$$

where $\theta \equiv \mu - \nu$, $U(\underline{x}, \underline{y}) \equiv \sum_{j=1}^n y_j$, $\xi \equiv -\mu$, and $T(\underline{x}, \underline{y}) \equiv \sum x_i + \sum x_j$. This rewrite works for testing the hypotheses (a) and (b) of problem 1, and the form of the tests remains the same as before with the new U and T, and all that remains is to calculate the conditional distributions of U given T. In (b) it is easily found that $\mu U \sim \text{Gamma}(n, 1)$, $\mu T \sim \text{Gamma}(m+n, 1)$, and hence $V \equiv U/T \sim \text{Beta}(n, m)$, so the test can be carried out unconditionally using tables of the Beta distributions.
B. The methods used above work for hypotheses of the form $H : g(\mu, \nu) \le 0$ versus $K : g(\mu, \nu) > 0$ and $g_1(\mu, \nu) = a\mu + b\nu + c$ for constants a, b, c with at least one of $a, b \neq 0$. Note that (a) above corresponds to $a = 1$, $b = -1$, $c = 1$ (one sided), (b) above corresponds to $a = 1$, $b = -1$, $c = 0$ (two-sided), and (c) above corresponds to $a = 1$, $b = -2$, $c = 0$ (two-sided). The above methods also work for $g_2(\mu, \nu) = a(\mu/\nu) - b$ (note that $\theta \equiv a(\mu/\nu) - b \le \theta_0$ if and only if $a\mu - b\nu \le \theta_0 \nu$ or equivalently if and only if $a\mu - (b + \theta_0)\nu \le 0$, so this is really equivalent to the situation for g_1) but they fail for $g_3(\mu, \nu) = a\mu\nu - b$.

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, and consider testing $H : \mu \le 0$ versus $K : \mu > 0$. Let $T_1 = \sum_{i=1}^n X_i$, $T_2 = \sum_{i=1}^n X_i^2$, and set $\underline{U} \equiv (X_1, \ldots, X_n)/\sqrt{T_2} \in R^n$.
A. Show that $|\underline{U}| = \sqrt{\sum_1^n U_i^2} = 1$, so $\underline{U}$ takes values in the unit sphere S^{n-1} in R^n. Show that when $\mu = 0$, $\underline{U} \sim \text{Uniform}(S^{n-1})$.
B. Let $\underline{d} \equiv n^{-1/2}\underline{1} = n^{-1/2}(1, \ldots, 1)$; note that it is a vector of length 1. Show that

$$\langle \underline{U}, \underline{d} \rangle = \frac{n^{-1}\sum_{i=1}^n X_i}{\sqrt{T_2/n}} .$$

Interpret the inequality $\langle \underline{U}, \underline{d} \rangle \le c$ as an event on the sphere S^{n-1}.
C. Find the distribution of $Y_n \equiv \langle \underline{U}, \underline{d} \rangle$ under $\mu = 0$. For fixed $\alpha \in (0, 1/2)$, determine $c = c_{n,\alpha}$ so that $P_0(Y_n > c_{n,\alpha}) = \alpha$.
D. Show that the test $\phi(T_1, T_2) = 1\{Y_n > c_\alpha\}$ is equivalent to the (one-sided) one sample $t-$test $\varphi(T_1, T_2) = 1\{\sqrt{n}\overline{X}_n/S_n > t_{n-1,\alpha}\}$ where $\overline{X}_n = T_1/n$, $S_n^2 = (T_2 - T_1^2/n)/(n-1)$, and $t_{n-1,\alpha}$ is the upper α quantile of the $t-$distribution with $n - 1$ degrees of freedom.

Solution: A. The first part is easy: since $\underline{U} \equiv (X_1, \ldots, X_n)/\sqrt{\sum_{i=1}^n X_i^2}$, it follows immediately that

$$|\underline{U}|^2 = \sum_{i=1}^n U_i^2 = \frac{\sum_1^n X_i^2}{\sum_1^n X_i^2} = 1 .$$

Thus $\underline{U}$ takes values in the unit sphere S^{n-1} in R^n. Since under $\mu = 0$, the joint density of $X_1, \ldots, X_n$ is given by

$$p(\underline{x}, \sigma) = (2\pi\sigma^2)^{-n/2} \exp(-\sum_{i=1}^n x_i^2/(2\sigma^2)) ,$$

it is clear that on each surface $\sum_{i=1}^n x_i^2 = t_2$, the joint density is constant, and hence the conditional distribution $(\underline{X}|T_2)$ is uniform on the sphere of radius $\sqrt{T_2}$. Thus $\underline{U} = (\underline{X}/\sqrt{T_2}|T_2)$ is uniform on the unit sphere S^{n-1}, and since this does not depend on T_2, $\underline{U}$ is independent of T_2. (This also follows from Basu's theorem: T_2

is sufficient and complete for σ^2, and $\underline{U}$ is ancillary, so $\underline{U}$ and T_2 are independent.)
B. Again the first part is easy:

$$\langle \underline{U}, \underline{d} \rangle = \sum_{i=1}^{n} \frac{X_i}{\sqrt{T_2}} n^{-1/2} = \frac{n^{-1}\sum_{i=1}^{n} X_i}{\sqrt{T_2/n}},$$

and this is $\cos(\Theta_n)$ where Θ_n is the (random) angle between $\underline{U}$ and $\underline{d}$. Since $\cos(\theta)$ is 1 for $\theta = 0$, and decreases to 0 when $\theta = \pi/2$, and then continues decreasing to -1 as θ continues increasing to π, the event $\langle \underline{U}, \underline{d} \rangle \le c$ is equivalent to $\underline{U}$ falling *outside* a spherical cap of angle $\arccos(c)$ centered at $\underline{d}$.
C. To find the distribution of $Y_n = \langle \underline{U}, \underline{d} \rangle$ under $\mu = 0$, I will proceed backward from the known distribution of the usual $t-$statistic, assuming $n \ge 2$. Let $T_1 \equiv \sum_1^n X_i$, $T_2 \equiv \sum_1^n X_i^2$. Then

$$\begin{aligned} \tau_n &\equiv \frac{\sqrt{n}\overline{X}_n}{S_n} = \frac{T_1\sqrt{n-1}}{(nT_2 - T_1^2)^{1/2}} \\ &= \frac{(T_1/\sqrt{T_2})\sqrt{n-1}}{(n - T_1^2/T_2)^{1/2}} \\ &= \frac{\sqrt{n}Y_n\sqrt{n-1}}{(n - nY_n^2)^{1/2}} = \frac{Y_n\sqrt{n-1}}{(1 - Y_n^2)^{1/2}}. \end{aligned} \tag{0.1}$$

Thus we have

$$\frac{Y_n^2}{1 - Y_n^2} = \frac{\tau_n^2}{n-1} \qquad \text{and} \qquad Y_n^2 = \frac{\tau_n^2/(n-1)}{1 + \tau_n^2/(n-1)}.$$

It follows that

$$\begin{aligned} P(Y_n > y) &= P\left(\frac{\tau_n/(n-1)^{1/2}}{\sqrt{1 + \tau_n^2/(n-1)}} > y \right) \\ &= P\left(\tau_n > \frac{y\sqrt{n-1}}{\sqrt{1-y^2}} \right) \\ &= 1 - F_{\tau_n}\left(\frac{y\sqrt{n-1}}{(1-y^2)^{1/2}} \right). \end{aligned}$$

Hence the density of Y_n is given by

$$f_{Y_n}(y) = f_{\tau_n}\left(\frac{y\sqrt{n-1}}{(1-y^2)^{1/2}} \right) \frac{\sqrt{n-1}}{(1-y^2)^{3/2}}.$$

Since the density of τ_n is the t_{n-1} density given by

$$f_{\tau_n}(t) = \frac{\Gamma(n/2)}{\sqrt{\pi(n-1)}\Gamma((n-1)/2)} \frac{1}{\left(1 + \frac{t^2}{n-1}\right)^{n/2}} \qquad -\infty < t < \infty,$$

it follows that

$$f_{Y_n}(y) = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n-1)/2)} (1-y^2)^{(n-3)/2}, \qquad -1 < y < 1.$$

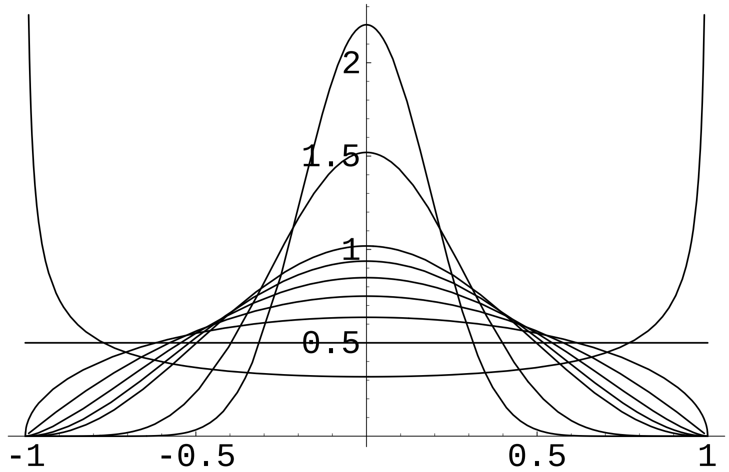

Figure 1: Density of Y_n, $n = 2 - 8(1), 16, 32$.

See Figure 1 for some plots of the densities f_{Y_n}. Thus the critical points $c_{n,\alpha}$ for Y_n should be chosen to satisfy

$$\int_{c_{n,\alpha}}^{1} \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n-1)/2)}(1-y^2)^{(n-3)/2} dy = \alpha \, .$$

D. The equivalence of the test ϕ given by $\phi(T_1, T_2) = 1\{Y_n > c_{n,\alpha}\}$ to the (one-sided) one sample $t-$test $\varphi(T_1, T_2) = 1\{\sqrt{n}\overline{X}_n / S_n > t_{n-1,\alpha}\}$ where $\overline{X}_n = T_1/n$, $S_n^2 = (T_2 - T_1^2/n)/(n-1)$, follows from the preceding arguments: note that from the identity (0.1) τ_n is an increasing function of Y_n, so large values of Y_n correspond to large values of τ_n.

4. A. Suppose that $X_1, \ldots, X_m$ be i.i.d. $N(\mu, \sigma^2)$ and $Y_1, \ldots, Y_n$ be i.i.d. $N(\nu, \sigma^2)$ are independent. Let $N \equiv m + n$. Consider testing $H : \mu = \nu$ versus $K : \mu \neq \nu$. Show that the usual two-sample $t-$ test, reject H if $|\tau_{m,n}| > t_{m+n-2,\alpha/2}$ where

$$\tau_{m,n} \equiv \frac{\sqrt{mn/N}(\overline{X}_m - \overline{Y_n})}{S_p}$$

and

$$S_p^2 \equiv \frac{1}{m+n-2}\left\{\sum_{i=1}^{m}(X_i - \overline{X}_m)^2 + \sum_{j=1}^{m}(Y_j - \overline{Y}_n)^2\right\} ,$$

is the UMP unbiased test of H versus K.
B. What happens if the variances in the two populations are different? What compromise test would you use?
C. How would you approximate the power of the tests in A and B for $\mu \neq \nu$ and if the underlying distributions are not necessarily normal?

Solution: A. This is nicely written out on pages 201-203, Lehmann, TSH, with the slightly different notation $\xi = \mu$, $\eta = \nu$. Note the argument on page 203 using $W = a(T_1^*, T_2^*) + b(T_1^*, T_2^*)U^*$, a linear function of U^* for fixed T_1^*, T_2^*.
B. When the variances are σ^2 and τ^2 with $\sigma^2 \neq \tau^2$, the problem of testing $H : \mu = \nu$ versus $K : \mu \neq \nu$ is called the Bhrens - Fisher problem. Unfortunately, the exponential family methods developed in Section 6.2 do not work in this problem, and there is no UMPU test. Note, however, that $\overline{X}_m - \overline{Y}_n \sim N(0, \sigma^2/m + \tau^2/n)$ where σ^2 and τ^2 can be naturally estimated by $S_X^2 = (m-1)^{-1}\sum_1^m (X_i - \overline{X})^2$ and $S_Y^2 = (n-1)^{-1}\sum_1^n (Y_i - \overline{Y})^2$ respectively. Moreover, under the null hypothesis $\mu = \nu$, we have

$$Z_{m,n} \equiv \frac{\overline{X}_m - \overline{Y}_n}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}} \to_d N(0,1)\,; \tag{0.2}$$

hence the test which rejects if $|Z_{m,n}| > z_{\alpha/2}$ has approximate size α. A refinement of this is to refer $Z_{m,n}$ to a $t-$distribution with random number of degrees of freedom f where, with

$$R \equiv \frac{S_X^2/m}{S_Y^2/n}\,,$$

f is defined by

$$\frac{1}{f} = \frac{R^2}{1+R^2}\frac{1}{m-1} + \frac{1}{1+R^2}\frac{1}{n-1}\,.$$

This is called the Welch approximate $t-$test.
C. If we assume the variances are equal and use the two-sample $t - test$ derived in A, then we can write

$$\begin{aligned}\tau_{m,n} &= \frac{\sqrt{\frac{mn}{N}}(\overline{X}_m - \overline{Y}_n)}{S_p} \\ &= \frac{\sqrt{\frac{mn}{N}}(\overline{X}_m - \overline{Y}_n) - (\mu - \nu)}{S_p} + \frac{\sqrt{\frac{mn}{N}}(\mu - \nu)}{S_p}\end{aligned}$$

where the first term converges in distribution to $Z \sim N(0,1)$, and $S_p^2 \to_p \sigma^2$. Thus, with $\delta_{m,n} \equiv \sqrt{mn/N}(\mu - \nu)/\sigma$,

$$\begin{aligned}&P_{\mu,\nu}\left(|\tau_{m,n}| \ge t_{m+n-2,\alpha/2}\right) \\ &= P_{\mu,\nu}\left(\left|\frac{\sqrt{\frac{mn}{N}}(\overline{X}_m - \overline{Y}_n - (\mu - \nu)}{S_p} + \frac{\sqrt{\frac{mn}{N}}(\mu - \nu)}{S_p}\right| > t_{m+n-2,\alpha/2}\right) \\ &\approx P\left(|Z + \delta_{m,n}| > t_{m+n-2,\alpha/2}\right)\,.\end{aligned}$$

On the other hand, if we do not assume that the variances are equal and use the compromise approximate $t - test$ in B, then since $S_X^2 \to_p \sigma^2$ and $S_Y^2 \to_p \tau^2$,

$$\begin{aligned}Z_{m,n} &\equiv \frac{\overline{X}_m - \overline{Y}_n}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}} \\ &= \frac{\overline{X}_m - \overline{Y}_n - (\mu - \nu)}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}} + \frac{\mu - \nu}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}}\end{aligned}$$

where the first term converges in distribution to $Z \sim N(0,1)$ and the second term is approximated by

$$\frac{\mu - \nu}{\sqrt{\frac{\sigma^2}{m} + \frac{\tau^2}{n}}} \equiv \tilde{\delta}_{m,n} .$$

Hence the power of the approximate $t-$test is

$$\begin{aligned}
&P_{\mu,\nu}\left(|Z_{m,n}| \geq t_{f,\alpha/2}\right) \\
&= P_{\mu,\nu}\left(\left|\frac{\overline{X}_m - \overline{Y}_n - (\mu - \nu)}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}} + \frac{\mu - \nu}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}}\right| > t_{f,\alpha/2}\right) \\
&\approx P\left(|Z + \tilde{\delta}_{m,n}| > z_{\alpha/2}\right) .
\end{aligned}$$