Statistics 583, Problem Set 1 Solutions

Wellner; 4/5/2000

1. Let $X_1, \ldots, X_n$ be a sample from the normal distribution $N(\mu, \sigma^2)$.
(i) If $\sigma = \sigma_0$ (known), there exists a UMP test for testing $H : \mu \le \mu_0$ against $K : \mu > \mu_0$, which rejects when $\sum_{i=1}^n (X_i - \mu_0)$ is "too large". Show this, and determine exactly what "too large" means.
(ii) If $\mu = \mu_0$ (known), there exists a UMP test for testing $H : \sigma \le \sigma_0$ against $K : \sigma > \sigma_0$ which rejects when $\sum_{i=1}^n (X_i - \mu_0)^2$ is "too large". Show this, and determine exactly what "too large" means.
(iii) Determine the asymptotic size of the UMP tests in (i) and (ii) when the X_i's fail to be normal, but are from a distribution F with $E_F(X^2) < \infty$ in (i) and $E_F(X^4) < \infty$ in (ii).

Solution: (i) First, when $\sigma = \sigma_0$ (known), we may reduce (by sufficiency) to $\sum_{i=1}^n X_i$, or equivalently to $\overline{X}_n \sim N(\mu, \sigma_0^2/n)$, or equivalently to

$$\sqrt{n}(\overline{X}_n - \mu_0)/\sigma_0 \sim N\left(\frac{\sqrt{n}(\mu - \mu_0)}{\sigma_0}, 1\right) \equiv N(\delta, 1)\,.$$

The resulting family of densities, namely $\{p(\cdot; \delta) : \ \delta \in R\}$ with $p(x;\delta) = \phi(x - \delta)$, is an exponential family with monotone likelihood ratio:

$$p(x;\delta) = c(\delta)\exp(\delta x) h(x)$$

where

$$c(\delta) \equiv \frac{1}{\sqrt{2\pi}}\exp(-\delta^2/2)\,, \qquad h(x) \equiv \exp(-x^2/2)\,,$$

and $Q(\delta) \equiv \delta$. Hence, by the Karlin-Rubin Theorem, the UMP test of $H : \mu \le \mu_0$ against $K : \mu > \mu_0$, or equivalently $H : \delta \le 0$ versus $K : \delta > 0$, is the test φ given by

$$\varphi(\overline{X}_n) = \begin{cases} 1 & \text{if } \sqrt{n}(\overline{X}_n - \mu_0)/\sigma_0 > C, \\ 0 & \text{if } \sqrt{n}(\overline{X}_n - \mu_0)/\sigma_0 \le C\,. \end{cases}$$

By choosing $C = z_\alpha$ satisfying $P(Z > z_\alpha) = \alpha$ (where $Z \sim N(0,1)$), it follows that $E_{\mu_0}\varphi(\overline{X}_n) = \alpha$, and hence φ is the UMP level α test.

(ii) When $\mu = \mu_0$ is known, we may reduce by sufficiency to $\sum_{i=1}^n (X_i - \mu_0)^2 \sim \sigma^2\chi_n^2$, or equivalently to

$$T(\underline{X}) \equiv \sum_{i=1}^n (X_i - \mu_0)^2/\sigma_0^2 \sim \frac{\sigma^2}{\sigma_0^2}\chi_n^2 \equiv \theta\chi_n^2\,.$$

The resulting family of densities, namely $\{p(\cdot;\theta) : \ \theta \in (0,\infty)\}$ with

$$p(x;\theta) = \frac{1}{2\theta}\left(\frac{x}{2\theta}\right)^{n/2-1}\exp(-x/(2\theta))\,1_{(0,\infty)}(x)\,,$$

is an exponential family with monotone likelihood ratio:

$$p(x;\theta) = c(\theta)\exp(Q(\theta)x)h(x)$$

where $c(\theta) = (2\theta)^{-n/2}$, $Q(\theta) = -(2\theta)^{-1}$, and $h(x) = x^{n/2-1}$. Hence, by the Karlin-Rubin Theorem, the UMP test of $H : \sigma \le \sigma_0$ against $K : \sigma > \sigma_0$, or equivalently $H : \theta \le 1$ versus $K : \theta > 1$, is the test φ given by

$$\varphi(\underline{X}) = \begin{cases} 1 & \text{if } T(\underline{X}) > C\,, \\ 0 & \text{if } T(\underline{X}) \le C\,. \end{cases}$$

By choosing $C = \chi^2_{n,\alpha}$ satisfying $P(\chi^2_n > \chi^2_{n,\alpha}) = \alpha$ using tables of the chi-square distribution, it follows that $E_{\mu_0}\varphi(\underline{X}) = \alpha$, and hence φ is the UMP level α test.

(iii). If $E_F(X^2) < \infty$, then under $E_F(X) = \mu_0$ and assuming that $Var_F(X) = \sigma_0^2$ is known, the (ordinary, Lindeberg) CLT yields

$$\sqrt{n}(\overline{X} - \mu_0)/\sigma_0 \to_d Z \sim N(0,1)\,.$$

Hence

$$P(\sqrt{n}(\overline{X}_n - \mu_0)/\sigma_0 > z_\alpha) \to P(Z > z_\alpha) = \alpha\,,$$

so this test has asymptotically correct size. (Note that this continues to be true if we had estimated the variance and used the normal - theory $t-$ statistic with critical points $t_{n-1,\alpha}$ from the $t-$distribution: this follows from $t_{n-1,\alpha} \to z_\alpha$ and Slutsky's theorem.)
If $E_F(X^4) < \infty$, then, under $\sigma = \sigma_0^2$, the random variables $Y_i \equiv (X_i - \mu_0)^2/\sigma_0^2$ have mean 1 and variance

$$Var(Y_i) = E\left\{[(X_i - \mu_0)^2/\sigma_0^2]^2\right\} - 1 = 2 + \gamma_2(F)$$

where

$$\gamma_2(F) \equiv E\left(\frac{(X_i - \mu_0)}{\sigma_0}\right)^4 - 3$$

is the (excess of) kurtosis. Note that for $X \sim F = N(\mu_0, \sigma_0^2)$, $\gamma_2(F) = 0$. Hence by the (classical or Lindeberg) CLT,

$$\frac{\sqrt{n}}{\sqrt{2}}\left(n^{-1}T(\underline{X}) - 1\right) = \frac{\sqrt{n}}{\sqrt{2}}\left(\overline{Y}_n - 1\right) \to_d N(0, 1 + \gamma_2(F)/2)\,.$$

Since $\sqrt{n/2}(n^{-1}\chi^2_{n,\alpha} - 1) \to z_\alpha$ (by the CLT applied to a chi-square random variable χ^2_n), it follows that

$$\begin{aligned} &P_{\sigma_0}(T(\underline{X}) > \chi^2_{n,\alpha}) \\ &\quad = P_{\sigma_0}\left(\sqrt{n/2}(n^{-1}T(\underline{X}) - 1) > \sqrt{n/2}(n^{-1}\chi^2_{n,\alpha} - 1)\right) \\ &\quad \to P(N(0, 1 + \gamma_2(F)/2) > z_\alpha) \ne \alpha \end{aligned}$$

when $\gamma_2(F) \ne 0$. In fact the asymptotic size of the test is $1 - \Phi(z_\alpha/\sqrt{1 + \gamma_2(F)/2})$ which varies from 0 to 1/2 as $\gamma_2(F)$ varies from -2 to ∞. (This result continues to hold if μ_0 is estimated by $\overline{X}_n$, and it carries over to all the normal-theory tests of variances.)

2. Let $X_1, \ldots, X_n$ be a sample from the uniform distribution on $(0, \theta)$ where $\theta > 0$.
(i) Let $X = (X_1, \ldots, X_n)$. For testing $H : \theta \le \theta_0$ against $K : \theta > \theta_0$, any test is UMP at level α for which $E_\theta \phi(X) = \alpha$ for $\theta \le \theta_0$ and $\phi(X) = 1$ when $X_{(n)} > \theta_0$.
(ii) For testing $H : \theta = \theta_0$ against the alternative $K : \theta \ne \theta_0$ a unique UMP test exists, and is given by

$$\phi(X) = \begin{cases} 1, & \text{if } X_{(n)} \le \theta_0 \alpha^{1/n} \text{ or } X_{(n)} > \theta_0 \\ 0, & \text{if otherwise}. \end{cases}$$

Solution: The sufficient statistic for θ is $T \equiv T_n \equiv \max_{1 \le i \le n} X_i \equiv X_{(n)}$, and T has the density $p(t; \theta) = n\theta^{-n} t^{n-1} 1_{[0,\theta]}(t)$.
(i) For testing $\theta = \theta_0$ versus $\theta = \theta_1 > \theta_0$, the class of all NP tests is given by tests of the form

$$\phi(t) = \begin{cases} 1 & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) > k\theta_0^{-n} 1_{[0,\theta_0]}(t) \\ \gamma(t) & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) = k\theta_0^{-n} 1_{[0,\theta_0]}(t) \\ 0 & \text{if } \theta_1^{-n} 1_{[0,\theta_1]}(t) < k\theta_0^{-n} 1_{[0,\theta_0]}(t). \end{cases}$$

Thus for $k = (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1 & \text{if } \theta_0 < t \le \theta_1 \\ \gamma(t) & \text{otherwise}; \end{cases} \tag{0.1}$$

for $k > (\theta_1/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1 & \text{if } \theta_0 < t \le \theta_1 \\ \gamma(t) & \text{otherwise}; \\ 0 & \text{if } 0 \le t \le \theta_0; \end{cases}$$

and for $k < (\theta_0/\theta_1)^n$ the NP tests are of the form

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \le t \le \theta_1 \\ \gamma(t) & \text{otherwise}; \end{cases}$$

The subclass of these tests that do not depend on θ_1 is the class of tests ϕ with

$$\phi(t) = \begin{cases} 1 & \text{if } \theta_0 < t < \infty \\ \gamma'(t) & \text{otherwise}; \end{cases} \tag{0.2}$$

where $E_{\theta_0} \gamma'(T) = \alpha$. In particular, we can get these tests from those in (0.1) by choosing

$$\gamma(t) = \begin{cases} 1 & \text{if } \theta_1 < t < \infty \\ \gamma'(t) & 0 \le t \le \theta_0 \\ 0 & \text{otherwise}. \end{cases}$$

In particular, the test $\phi_1(t) = 1\{t > \theta_0\} + \alpha 1\{t \le \theta_0\}$ is of the form of the tests in (0.2) with $\gamma(t) = \alpha$ for $0 \le t \le \theta_0$, and hence is UMP of size α for testing $\theta = \theta_0$ versus $\theta > \theta_0$. To see that it is of size α for the composite null hypothesis $\theta \le \theta_0$,

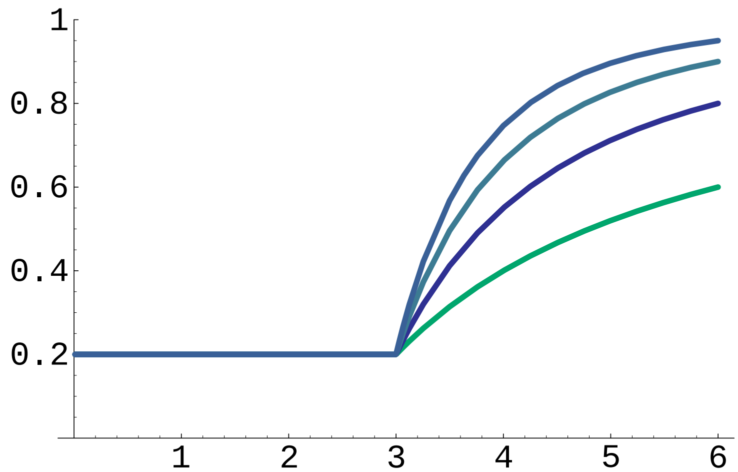

Figure 1:Power of ϕ_1 when $\theta_0 = 3$, $\alpha = .2$, $n = 1, 2, 3, 4$.

we first compute its power function to confirm that it is of size α for the composite null hypothesis $\theta \le \theta_0$. The power is:

$$\begin{aligned}
\beta_{\phi_1}(\theta) &= E_\theta \phi(T) = P_\theta(T > \theta_0) + \alpha P_\theta(T \le \theta_0) \\
&= \left\{1 - \left(\frac{\theta_0}{\theta}\right)^n\right\} 1_{(\theta_0,\infty)}(\theta) + \alpha 1_{[0,\theta_0]}(\theta) + \alpha(\theta_0/\theta)^n 1_{(\theta_0,\infty)}(\theta)\,, \\
&= \left\{1 - \left(\frac{\theta_0}{\theta}\right)^n (1-\alpha)\right\} 1_{(\theta_0,\infty)}(\theta) + \alpha 1_{[0,\theta_0]}(\theta)\,.
\end{aligned}$$

Thus we see that $\beta_{\phi_1}(\theta) = \alpha$ for $\theta \le \theta_0$, so that $\sup_{\theta\in\Theta_0} E_\theta \phi_1(X) = \alpha$. Since the class of size α tests for testing $\theta = \theta_0$ is a larger class than the class of size α tests for testing $\theta \le \theta_0$, and since ϕ_1 is UMP in the larger class, it is also UMP in the smaller class. Hence ϕ_1 is UMP for testing $\theta \le \theta_0$ versus $\theta > \theta_0$. But the competing test $\phi_0(t) = 1\{t > (1-\alpha)^{1/n}\theta_0\}$ has power function

$$\begin{aligned}
\beta_{\phi_0}(\theta) &= E_\theta \phi_0(T) = P_\theta(T > \theta_0(1-\alpha)^{1/n}) \\
&= 1 - \left\{(1-\alpha)\left(\frac{\theta_0}{\theta}\right)^n 1_{(\theta_0(1-\alpha)^{1/n},\infty)}(\theta) + 1_{[0,\theta_0(1-\alpha)^{1/n}]}(\theta)\right\}\,,
\end{aligned}$$

so the power function of the test ϕ_0 is strictly below that of the test ϕ_1 on the set $[0, \theta_0)$. Hence ϕ_1 is inadmissible, and the test ϕ_0 is also UMP. (See Figures 1 and 2.)

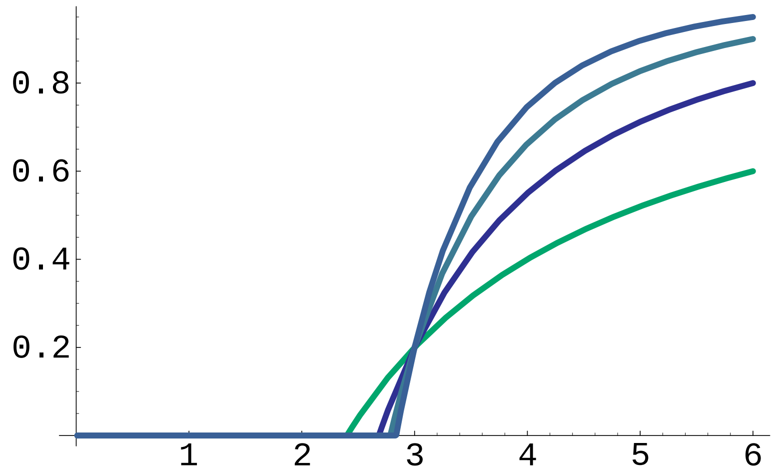

Figure 2:Power of ϕ_0 when $\theta_0 = 3$, $\alpha = .2$, $n = 1, 2, 3, 4$.

(ii) The test $\phi_2(t) = 1 - 1_{(\theta_0 \alpha^{1/n}, \theta_0]}(t)$ is of size α for testing $\theta = \theta_0$ versus $\theta \neq \theta_0$ since

$$E_{\theta_0}\phi_2(T) = P_{\theta_0}(T \le \theta_0 \alpha^{1/n}) = \left(\frac{\theta_0 \alpha^{1/n}}{\theta_0}\right)^n = \alpha\,.$$

Furthermore, it is of the form of the class of all UMP tests for testing $\theta = \theta_0$ versus $\theta > \theta_0$, and hence it is UMP among the size α tests for $\theta > \theta_0$. For testing $\theta = \theta_0$ versus $\theta = \theta_1 < \theta_0$, the Neyman-Pearson tests of the form $\phi_3(t) = \gamma(t) 1_{[0,\theta_0]}(t)$ are most powerful of their size. But the test ϕ_2 is of this form (with $\gamma(t) = 1_{[0,\theta_0\alpha^{1/n}]}(t)$), is in this class, and does not depend on $\theta_1 < \theta_0$. Hence ϕ_2 is UMP for testing $\theta = \theta_0$ versus $\theta \neq \theta_0$. The power function of the two-sided test ϕ_3 is given by

$$\begin{aligned}
\beta_{\phi_3}(\theta) &= E_\theta \phi_3(T) = P_\theta(T \le \theta_0 \alpha^{1/n}) + P_\theta(T > \theta_0) \\
&= \left(\frac{\theta_0 \alpha^{1/n}}{\theta}\right)^n 1\{\theta_0 \alpha^{1/n} \le \theta\} + 1\{\theta_0 \alpha^{1/n} > \theta\} \\
&\qquad + \ (1 - (\theta_0/\theta)^n)\, 1\{\theta_0 \le \theta\} \\
&= \begin{cases} 1 & \text{if } \theta < \theta_0 \alpha^{1/n} \\ (\theta_0/\theta)^n \alpha & \text{if } \theta_0 \alpha^{1/n} \le \theta \le \theta_0 \\ 1 - (\theta_0/\theta)^n (1-\alpha) & \text{if } \theta_0 < \theta < \infty\,. \end{cases}
\end{aligned}$$

See Figure 3. [This is an unusual situation in which we get “something for free” from the structure of the uniform distributions. Usually two-sided tests are *not* UMP!]

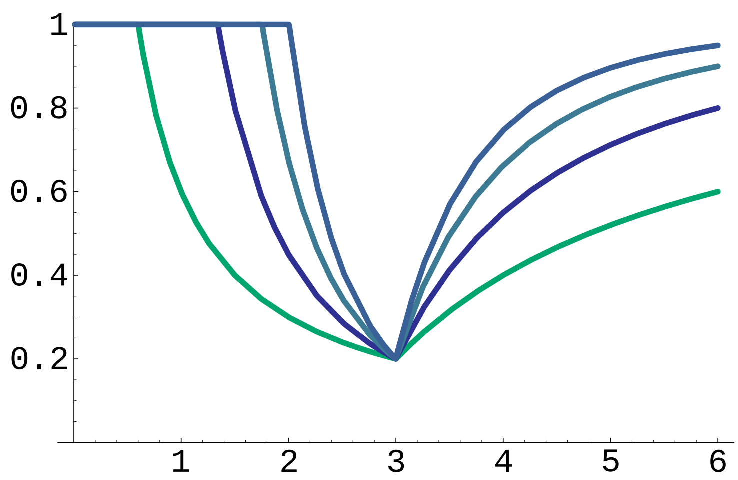

Figure 3:Power of ϕ_3 when $\theta_0 = 3$, $\alpha = .2$, $n = 1, 2, 3, 4$

3. Let X be the number of successes in n independent trials with probability p of success, and let ϕ be the UMP test

$$\phi(X) = \begin{cases} 1, & \text{if } X > C\,, \\ \gamma, & \text{if } X = C\,, \\ 0, & \text{if } X < C\,. \end{cases}$$

(i) For $n = 7$, $p_0 = .25$, and the levels $\alpha = 0.05$, 0.1, and 0.2, determine C and γ.
(ii) If $p_0 = .3$ and $\alpha = .05$, and it is desired to have power $\beta \geq 0.9$ against $p_1 = .4$, determine the necessary sample size:
(a) by using a table of the Binomial distribution;
(b) by using the normal approximation (Lehmann notes that tables an approximations are discussed in Chapter 3 of Johnson and Kotz (1969), but they can of course be found in many other places. What is the effect of correcting for continuity in using a normal approximation here?
(iii) Use the normal approximation to determine the sample size required when $\alpha = 0.05$, $\beta = .9$, $p_0 = 0.02$ and $p_1 = 0.04$.

Solution: (i) When $n = 7$ and $p_0 = .25$, we have the following distribution of X under p_0:

k	0	1	2	3	4	5	6	7
$P(X = k)$	.13348	.31146	.31146	.17304	.05768	.01154	.00128	.00006
$P(X > k)$	.8665	.5550	.2436	.0706	.0129	.0013	.00006	0

Thus:
for $\alpha = .05$ we would choose $c = 4$ and $\gamma = (.05 - .0129)/.05768 = .6432$;
for $\alpha = .10$ we would choose $c = 5$ and $\gamma = (.10 - .0706)/.17304 = .1699$; and
for $\alpha = .20$ we would choose $c = 5$ and $\gamma = (.20 - .0706)/.17304 = 0.7478$.

(ii) When $p_0 = .3$, $\alpha = .05$, and we want $\beta(.4) \geq .9$, we want to determine an integer C, $\gamma \in [0,1]$, and n so that both of the following hold:

$$P_{.3}(T_n > C) + \gamma P_{.3}(T_n = C) = .05\,,$$
$$P_{.5}(T_n > C) + \gamma P_{.4}(T_n = C) \geq .90\,.$$

(b) We will first do this approximately using normal approximation and ignoring the term involving γ. Now by using the normal approximation to the Binomial distribution (with correction for continuity),

$$\begin{aligned} P_{.3}(T_n > C) &= P_{.3}(T_n > C + .5) \\ &= P_{.3}\left(\frac{T_n - np_0}{\sqrt{np_0(1-p_0)}} > \frac{C + .5 - np_0}{\sqrt{np_0(1-p_0)}}\right) \\ &\doteq P\left(Z > \frac{C + .5 - np_0}{\sqrt{np_0(1-p_0)}}\right)\,. \end{aligned}$$

The right side is approximately .05 if

$$\frac{C + .5 - np_0}{\sqrt{np_0(1-p_0)}} = z_{.05} = 1.645\,;$$

or equivalently if

$$C = z_{.05}\sqrt{np_0(1-p_0)} - .5 + np_0\,.$$

Then the power is given approximately by

$$\begin{aligned} \text{Power} &= P_{.4}(T_n > C) = P_{.4}(T_n > C) \\ &= P_{.4}\left(\frac{T_n - np_1}{\sqrt{np_1(1-p_1)}} > \frac{z_\alpha\sqrt{np_0(1-p_0)} - n(p_1 - p_0)}{\sqrt{np_1(1-p_1)}}\right) \\ &\doteq P\left(Z > z_\alpha\sqrt{\frac{.3(1-.3)}{.4(1-.4)}} - \frac{\sqrt{n}(.1)}{\sqrt{p_1(1-p_1)}}\right) \end{aligned}$$

and the latter is $\geq .90$ if n is so large that

$$z_{.05}\sqrt{\frac{.3(1-.3)}{.4(1-.4)}} - \frac{\sqrt{n}(.1)}{\sqrt{p_1(1-p_1)}} \leq -1.2816$$

and hence

$$\sqrt{n} \geq \frac{(1.645\sqrt{(.3)(.7)/[(.4)(.6)]} + 1.2816)(\sqrt{(.4)(.6)})}{.1} = 13.82$$

or $n \geq 13.82^2 \doteq 191$. This yields

$$C = z_{.05}\sqrt{np_0(1-p_0)} - .5 + np_0 = 67\,.$$

(a) The exact probabilities (computed via Mathematica) are:

$$P_{.3}(T_{191} > 67) = .0553, \quad P_{.3}(T_{191} = 67) = .0193$$
$$P_{.3}(T_{191} > 68) = .0402, \quad P_{.3}(T_{191} = 68) = .0151\,.$$

Thus we could choose $C = 68$, $\gamma = .649$, and then

$$\begin{aligned} E_{.3}\phi(T) &= P_{.3}(T_{191} > 68) + \gamma P_{.3}(T_{191} = 68) \\ &= .0402 + .649 \times .0151 = .0499\,, \end{aligned}$$

while

$$\begin{aligned} \beta_\phi(.4) &= P_{.4}(T_{191} > 68) + .649 * P_{.4}(T_{191} = 68) \\ &= .8788 + .649 \times .0276 = 0.8967\,. \end{aligned}$$

If we increase the sample size by 1 to $n = 192$, then the exact probabilities are

$$P_{.3}(T_{192} > 68) = .0447, \quad P_{.3}(T_{192} = 68) = .0164$$

Thus we could choose $C = 68$, $\gamma = .3232$, and then

$$\begin{aligned} E_{.3}\phi(T) &= P_{.3}(T_{192} > 68) + \gamma P_{.3}(T_{192} = 68) \\ &= .0447 + .3232 \times .0164 = .0500\,, \end{aligned}$$

while

$$\begin{aligned} \beta_\phi(.4) &= P_{.4}(T_{192} > 68) + .3232 \times P_{.4}(T_{192} = 68) \\ &= .8898 + .3232 \times .0256 = 0.8981\,. \end{aligned}$$

(iii) When $p_0 = .02$, $\alpha = .05$, and we want $\beta(.04) \geq .9$, we want to use the CLT to determine an integer C, $\gamma \in [0, 1]$, and n so that both of the following hold:

$$\begin{aligned} P_{.02}(T_n > C) + \gamma P_{.02}(T_n = C) = .05\,, \\ P_{.04}(T_n > C) + \gamma P_{.04}(T_n = C) \geq .90\,. \end{aligned}$$

Again ignoring the term involving γ and using the normal approximation to the Binomial distribution (with correction for continuity),

$$\begin{aligned} P_{.02}(T_n > C) &= P_{.02}(T_n > C + .5) \\ &= P_{.02}\left(\frac{T_n - np_0}{\sqrt{np_0(1-p_0)}} > \frac{C + .5 - np_0}{\sqrt{np_0(1-p_0)}} \right) \\ &\doteq P\left(Z > \frac{C + .5 - np_0}{\sqrt{np_0(1-p_0)}} \right)\,. \end{aligned}$$

The right side is approximately .05 if

$$\frac{C + .5 - np_0}{\sqrt{np_0(1-p_0)}} = z_{.05} = 1.645\,;$$

or equivalently if

$$C = z_{.05}\sqrt{np_0(1-p_0)} - .5 + np_0\,.$$

Then the power is given approximately by

$$\begin{aligned}
\text{Power} &= P_{.04}(T_n > C) \\
&= P_{.04}\left(\frac{T_n - np_1}{\sqrt{np_1(1-p_1)}} > \frac{z_\alpha\sqrt{np_0(1-p_0)} - n(p_1 - p_0)}{\sqrt{np_1(1-p_1)}}\right) \\
&\doteq P\left(Z > z_\alpha\sqrt{\frac{.02(1-.02)}{.04(1-.04)}} - \frac{\sqrt{n}(.02)}{\sqrt{p_1(1-p_1)}}\right)
\end{aligned}$$

and the latter is $\geq .90$ if n is so large that

$$z_{.05}\sqrt{\frac{.02(1-.02)}{.04(1-.04)}} - \frac{\sqrt{n}(.02)}{\sqrt{p_1(1-p_1)}} \leq -1.2816$$

and hence

$$\sqrt{n} \geq \frac{(1.645\sqrt{(.02)(.98)/[(.04)(.96)]} + 1.2816)(\sqrt{(.04)(.96)})}{.02} = 24.07$$

or $n \geq 24.07^2 \doteq 579$. This yields

$$C = z_{.05}\sqrt{np_0(1-p_0)} - .5 + np_0 = 16\,.$$

Here the normal approximation has a hard time: the exact size and power of the resulting test are .078 and .927 respectively. If we change C to 17, then the size and power are .047 and .8847 respectively. In this situation, Poisson approximations are a natural alternative to the Normal theory approximations, but I won't pursue this further here.

4. Let

$$f(x;\theta,\sigma) = \frac{1}{2\sigma}\exp(-\frac{|x-\theta|}{\sigma})$$

be the double exponential density.
(i) Show that this family has monotone likelihood ratio for the parameter θ when σ is known.
(ii) Is this an "exponential family" of distributions? Does the monotone likelihood ratio property in (i) imply that the corresponding densities of a random sample $X_1, \ldots, X_n$ i.i.d. with density $f(\cdot;\theta,\sigma)$ have monotone likelihood ratio if σ is known?

Solution: (i) Suppose that σ is known. Let $\theta_0 < \theta_1$. Then we have

$$\frac{f(x;\theta_1,\sigma)}{f(x;\theta_0,\sigma)} = \exp\left\{\frac{1}{\sigma}\left(|x-\theta_0| - |x-\theta_1|\right)\right\},$$

which is a monotone increasing (non-decreasing) function of x. Thus the family has MLR for fixed σ.
(ii) This is *not* an exponential family of distributions for fixed σ. (Of course it *is* an exponential family of distributions when θ is known and σ is unknown.) There is no elementary argument which says that the MLR property carries over from $n = 1$ to larger sample sizes. I believe that the MLR property fails for all $n > 1$, but I have not shown this. As pointed out by several of you, Lehmann (TSH) says (page 550), in a discussion of "Conditional Inference" that:

> "The double-exponential and logistic distributions are both strongly unimodal (Section 9.2), and thus provide examples of UMP conditional tests. In neither case does there exist a UMP unconditional test unless $n = 1$."

This suggest that both the MLR property and existence of UMP tests fails in this case, but I have not (yet) seen a proof of either assertion.