Statistics 583, Problem Set 6

Wellner; 5/3/2000

Reading: Lecture Notes, Chapter 6, pages 25-42;
Lehmann, TSH (Second Ed.), Chapter 6, pages 282 - 364;
Ferguson, MS, Chapter 5, Sections 5.6-5.7, pages 242-257.

Due: Wednesday, May 10, 2000.

1. Suppose that X_{ijk}, $i = 1, \ldots, I$, $j = 1, \ldots, J$, $k = 1, \ldots, K$ satisfy the general linear model with $\xi_{ijk} = \xi + \mu_i + \eta_j + \delta_{ij}$ where $\sum_i \mu_i = 0$, $\sum_j \eta_j = 0$, $\sum_j \delta_{ij} = 0$ for all i, and $\sum_i \delta_{ij} = 0$ for all j. (δ_{ij} is called the interaction effect of the ith row and the jth column.)
(a) Show that

$$\begin{aligned} S^2 &= \sum\sum\sum (X_{ijk} - \xi - \mu_i - \eta_j - \delta_{ij})^2 \\ &= \sum\sum\sum (X_{ijk} - \overline{X}_{ij\cdot})^2 \\ &\quad + \sum\sum\sum (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdots} - \delta_{ij})^2 \\ &\quad + \sum\sum\sum (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdots} - \mu_i)^2 \\ &\quad + \sum\sum\sum (\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdots} - \eta_j)^2 \\ &\quad + \sum\sum\sum (\overline{X}_{\cdots} - \xi)^2 \end{aligned}$$

where $\overline{X}_{ij\cdot} = \sum_k X_{ijk}/K$, and so on.
(b) Find the UMP invariant test of the hypothesis of no row effect $H_0 : \mu_1 = \ldots = \mu_I = 0$. What is the distribution of the test statistic under the general linear hypothesis – including the noncentrality parameter?
(c) Find the UMP invariant test of the hypothesis of no interaction effect $H_0 : \delta_{ij} = 0$ for all i, j. What is the distribution of the test statistic under the general linear hypothesis?

2. Suppose that X_i, $i = 1, \ldots, I$ and Y_j, $j = 1, \ldots, J$ satisfy the general linear hypothesis with $E(X_i) = \alpha_1 + \beta_1 u_i$ and $E(Y_j) = \alpha_2 + \beta_2 v_j$ where the u_i and v_j are known and $\sum u_i = \sum v_j = 0$, $\sum u_i^2 = I$, $\sum v_j^2 = J$. Find the UMP invariant test of $H : \beta_1 = \beta_2$ versus $K : \beta_1 \neq \beta_2$. What is the noncentrality parameter δ^2 for your test?

3. What is the locally best rank test of $F = G$ against $F = (e^{\theta G} - 1)/(e^\theta - 1)$, $\theta > 0$? Of $F = G$ against $F = G/(e^\theta(1 - G) + G)$?
What can you say about the power of these tests?

4. Suppose that $X_1, \ldots, X_n$ are independent Exponential(1) random variables. Let $Y_i \equiv X_{(i)}$, for $i = 1, \ldots, n$, denote the *order statistics* corresponding to $X_1, \ldots, X_n$.
(a) Show that the vector $(Y_1, \ldots, Y_n)$ has the same joint distribution as $(W_1, \ldots, W_n)$ where $W_i \equiv \sum_{j=1}^{i} Z_j/(n-j+1)$ and $Z_1, \ldots, Z_n$ are i.i.d. Exponential(1).
(b) Use the result of (a) to compute $E(Y_i)$, $Var(Y_i)$, and $Cov(Y_i, Y_j)$ for any fixed i, j.